Number of solutions to linear systems and Rank

Chapter 7

In this additional document we discuss the number of solutions to linear systems \( Ax = b \) in the light of the rank of \( A \). Let \( A \in \mathbb{R}^{m \times n} \) and consider the linear system

\[
Ax = b
\]

for some right-hand side \( b \in \mathbb{R}^n \). Let \( \{ c_1, \ldots, c_n \} \) denote the columns of \( A \). Since

\[
\text{col}(A) = \text{span}\{ c_1, \ldots, c_n \} \subset \text{span}\{ c_1, \ldots, c_n, b \} = \text{col}([A|b])
\]

it always holds

\[
\text{rank}(A) = \dim(\text{col}(A)) \leq \dim(\text{col}([A|b])) = \text{rank}([A|b])
\]

Let us now review the number of solutions to (1). We will consider two cases: \( b \in \text{Im}(A) \) and \( b \notin \text{Im}(A) \).

- Assume that \( b \notin \text{Im}(A) \). Then we have the strict inclusion \( \text{col}(A) \subsetneq \text{col}([A|b]) \) so

\[
\text{rank}(A) < \text{rank}([A|b])
\]

- Assume that \( b \in \text{Im}(A) \). Clearly we have

\[
(1) \text{ admits at least one solution } \iff b \in \text{Im}(A) = \text{col}(A) \\
\iff \text{rank}(A) = \text{rank}([A|b])
\]

To proceed further we need to consider two subcases, depending on the dimension of the null space of \( A \). We will use the fact that the solution to (1) writes as

\[
x = x_0 + \xi
\]

where \( x_0 \) is a particular solution to (1) and \( \xi \in \text{null}(A) \).

- If \( \dim(\text{null}(A)) = 0 \) then there is a unique solution (\( x_0 \)) and

\[
\text{rank}(A) = \text{rank}([A|b]) = n
\]

from (4) and using the dimension formula \( \dim(\text{null}(A)) = n - \text{rank}(A) = 0 \).

- If \( \dim(\text{null}(A)) > 0 \) there exists an infinite number of solutions (\( \{ x = x_0 + \xi \} \)) and

\[
\text{rank}(A) = \text{rank}([A|b]) < n
\]

since \( \text{rank}(A) = n - \dim(\text{null}(A)) < n \).

In summary we have shown that

<table>
<thead>
<tr>
<th>Condition on the rank</th>
<th>Number of solutions</th>
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<tbody>
<tr>
<td>( b \notin \text{col}(A) )</td>
<td>( \text{rank}(A) &lt; \text{rank}([A</td>
</tr>
<tr>
<td>( b \in \text{col}(A) )</td>
<td>( \text{rank}(A) = \text{rank}([A</td>
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<tr>
<td>( \text{rank}(A) = n (\text{null}(A) = {0}) )</td>
<td>( \text{rank}(A) &lt; n (\dim(\text{null}(A)) &gt; 0) )</td>
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</table>

\[1\] Let \( x_0 \) be a particular solution to (1). Then \( x \) is solution to (1) if and only if \( x = x_0 + \xi \). Indeed, if \( x \) is solution to (1) then \( \xi := x - x_0 \) belongs to \( \text{null}(A) \). Conversely, if \( x = x_0 + \xi \) then \( Ax = A(x_0 + \xi) = b + 0 = b \).