A priori error estimates for finite element approximations of parabolic stochastic partial differential equations with generalized random variables

Christophe Audouze & Prasanth B. Nair*

University of Toronto Institute for Aerospace Studies, 4925 Dufferin Street, Ontario, Canada M3H 5T6

(Received 00 Month 200x; in final form 00 Month 200x)

Abstract

We consider finite element approximations of parabolic stochastic partial differential equations (SPDEs) in conjunction with the $\theta$-weighted temporal discretization scheme. We study the stability of the numerical scheme and provide a priori error estimates, using a result of Galvis and Sarkis [13] on elliptic stochastic partial differential equations.

Keywords: Stochastic partial differential equations; White noise analysis; Time-stepping stability; A priori error estimation; Finite element method

AMS Subject Classification: 60H15; 60H40; 65M12; 65M60

1. Introduction

Stochastic partial differential equation (SPDE) models arise in a number of important application areas in applied sciences and engineering, including uncertainty quantification, design optimization and inverse parameter estimation. A variety of numerical schemes for solving SPDEs can be found in the literature; for example, polynomial chaos (PC) based stochastic projection schemes [1], multi-element generalized PC methods [2], generalized spectral decomposition schemes [3–6] and stochastic collocation procedures [7]. In PC projection schemes, the SPDE solution is typically approximated using a tensor product of compactly supported finite element basis functions and global PC basis, and the undetermined coefficients are computed using a stochastic Galerkin or Petrov-Galerkin projection scheme. Over the last two decades, this approach has been applied with much success to a broad range of SPDEs.

More recently, a number of researchers have studied the theoretical properties of stochastic PC projection schemes applied to SPDEs. In [8], some error estimates in tensor product Sobolev spaces are presented for a steady-state stochastic
diffusion model. These estimates provide \textit{a priori} convergence rates for fixed number of random variables but it is not possible to prove convergence for the case when the number of random variables goes to infinity. This case is of particular interest when random field coefficients of SPDE models are discretized using a Karhunen-Loève expansion scheme \cite{19}. In \cite{22}, convergence analysis in tensor product Sobolev spaces is conducted for the time-dependent stochastic diffusion model when considering stochastic Galerkin and stochastic collocation schemes, without considering the effect of temporal discretization errors. As mentioned in \cite{22}, a suitable choice of temporal discretization schemes leading to stable algorithms is a crucial task in the context of stochastic finite element approximations of time-dependent SPDEs. Nistor and Schwab \cite{21} presented error estimates for solutions of elliptic SPDEs that belong to Babuška-Kondratiev spaces. The error estimates provided therein depend on the norm of the source term and on the number of finite element degrees of freedom. Some convergence rates with respect to Sobolev norms are also given in \cite{29} for the steady-state stochastic diffusion model when considering sparse Weiner-chaos approximations.

In \cite{14}, a stochastic diffusion equation with a log-normal random field is studied in the \textit{infinite-dimensional noise} (also referred to as \textit{white noise}) framework. An \textit{a priori} error estimate in chaos weighted norms is provided for the stochastic Galerkin projection scheme. An attractive feature of the infinite-dimensional noise setting is that it allows us to deal with singular SPDE solutions, for example, solutions which may not have finite variance.

Stochastic distribution spaces have been used in \cite{9, 11, 13} to derive error estimates for finite element approximations of elliptic SPDEs with infinite-dimensional noise. It was shown that by carrying out analysis in the so-called Kondratiev space of stochastic distributions (which means we are taking stochastic regularity of the solution into account), it becomes possible to derive an \textit{a priori} estimate of the convergence rate. Theting \cite{25, 26} has rigorously studied Wick-type linear elliptic and parabolic SPDEs in the framework of stochastic distribution spaces and \textit{a priori} convergence rates are derived for various cases.

This paper is concerned with the derivation of \textit{a priori} error estimates for finite element approximations of parabolic SPDEs in the framework of stochastic distribution spaces. We consider classical-type (in contrast to Wick-type) parabolic SPDEs. In the present analysis, we consider a $\theta$-weighted temporal discretization procedure and study conditions under which this scheme is stable. Using the stability analysis results in conjunction with a result of Galvis and Sarkis \cite{13}, we provide \textit{a priori} error estimates for finite element approximations of parabolic SPDEs.

\section{Preliminaries}

First, consider the spatial domain $\mathcal{D}$ which is an open, connected, bounded convex subset of $\mathbb{R}^d$ with polygonal boundary $\partial \mathcal{D}$, and $[0,T]$ with $T < +\infty$ is the time interval over which the SPDE solution is sought. Next, let us introduce the mathematical background and notations associated with the infinite-dimensional noise setting (see \cite{9, 11, 16, 25, 26}). We denote by $S(\mathbb{R}^p)$ and $S'(\mathbb{R}^p)$ ($S$ and $S'$ in brief) the Schwartz space of smooth and rapidly decreasing functions on $\mathbb{R}^p$ and its dual space, the space of tempered distributions, respectively. The Bochner-Minlos theorem ensures that there exists a unique probability measure $\mu$ defined on $\mathcal{B}(S')$, where $\mathcal{B}(S')$ is the $\sigma$-field of Borel subsets of $S'$ with respect to the weak-star topol-
The probability measure $\mu$ referred to as the Gaussian white noise probability measure satisfies

$$\int_S e^{i\langle \omega, \phi \rangle} d\mu(\omega) = e^{-\frac{1}{2} ||\phi||^2_{L^2(\mathbb{R}^p)}}, \forall \phi \in S,$$

where $\langle \cdot, \cdot \rangle$ represents the duality pairing on $S' \times S$ (see [16] for more details). The white noise probability space is denoted by the triplet $(S', B(S'), \mu)$.

In this work, we consider parabolic SPDEs of the form

$$\frac{\partial u(x, t; \omega)}{\partial t} + Lu(x, t; \omega) = f(x, t; \omega) \text{ \mu-a.e. in } D \times [0, T] \times S',$$
$$u(x, t; \omega) = g(x; \omega) \text{ \mu-a.e. on } \partial D \times [0, T] \times S',$$
$$u(x, 0; \omega) = u_0(x; \omega) \text{ \mu-a.e. on } D \times S',$$

where $L$ is a linear second-order operator with respect to the spatial coordinates that is independent of time. The SPDE solution as a function of the spatial coordinates (resp. the temporal variable) is sought in the Hilbert space $V_x$ (resp. $V_t$). For simplicity of presentation we consider the case when $g(x; \omega) = 0$, however, the present analysis can be easily extended to non-homegeneous Dirichlet boundary conditions. Typically, $V_x$ denotes the classical Sobolev space $H^1_0(D)$ defined as the subspace of $H^1(D)$ consisting of functions which vanish on $\partial D$ in the sense of trace with $||w||^2_{H^1_0(D)} := \int_D |\nabla w(x)|^2 dx$, while $V_t = L^2(0, T)$ represents the space of square integrable functions over $[0, T]$. Henceforth, we shall denote by $H_1 \otimes H_2$ the tensor product of two Hilbert spaces $H_1$ and $H_2$.\(^1\)

Additional concepts and notations need to be introduced to define a suitable random function space $V_\omega$.\(^2\) Consider the multi-indices notation $\alpha = (\alpha_1, \alpha_2, \ldots)$, $\alpha_j \in \mathbb{N} = \{0, 1, 2 \ldots \}$ and define the set

$$\mathcal{J} = \{ \alpha, |\alpha| = \sum_j \alpha_j < +\infty \}$$

and the index $l(\alpha) = \max\{j \in \mathbb{N}, \alpha_j \neq 0 \}$. An orthogonal basis for the space of square integrable functions with respect to the measure $\mu$ (denoted by $L^2(\mu)(S')$) can be defined as

$$H_\alpha(\omega) = \prod_{j=1}^{l(\alpha)} h_{\alpha_j}(\langle \omega, \xi_j \rangle), \omega \in S', \alpha \in \mathcal{J},$$

where $h_{\alpha_j}$ denotes one-dimensional Hermite polynomials of degree $\alpha_j$ and $(\xi_j)_{j=1}^\infty$ is an orthonormal basis of $L^2(\mathbb{R}^p)$ obtained by using tensor products of one-dimensional Hermite functions in $L^2(\mathbb{R})$. In particular, note that $\xi_j \in S(\mathbb{R}^p)$. For

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\(^1\)As an example, consider $H_1 = V_x$, $H_2 = V_t$ and finite summations $u(x, t) = \sum_i u_{1,i}(x)u_{2,i}(t)$ with $u_{1,i} \in V_x$, $u_{2,i} \in V_t$. The Hilbert tensor product space $V_x \otimes V_t$ is defined as the closure of these sums with respect to the inner product $[u, v]_{V_x \otimes V_t} = \sum_{i,j} (u_{1,i}, v_{1,j})_{V_x}(u_{2,i}, v_{2,j})_{V_t}$; for example, see [8].

\(^2\)In the finite-dimensional noise setting, the probability space is the triplet $(\Gamma, B(\Gamma), \rho(\xi))$ where $\Gamma$ is the joint image of a vector $\xi : \Omega \rightarrow \mathbb{R}^M$ of uncorrelated random variables defined on a sample space $\Omega$, $B(\Gamma)$ is the $\sigma$-algebra associated with $\Gamma$ and $\rho$ is the joint probability density function. The random function space is usually defined as $V_\xi = L^2(\Gamma)$. 
We have the following (unique) Wiener-Itô chaos expansion
\[ f = \sum_{\alpha \in J} f_{\alpha} H_{\alpha}, \]
with
\[ |f|_{\rho,q,V_x}^2 = \sum_{\alpha \in J} |f_{\alpha}|_{V_x(\alpha!)}^2 (2N^\rho)_{\alpha}, \]
and
\[ \left( \alpha! \right)^{1+\rho} = \prod_{j=1}^{2N^\rho} (\alpha_j!)^{1+\rho}. \]

We are now in a position to define the so-called Kondratiev stochastic distribution spaces that will be used to derive error estimates for finite element approximations of parabolic SPDEs. Given \( \rho \in [-1,1], q \in \mathbb{R} \) and \( f = \sum_{\alpha \in J} f_{\alpha} H_{\alpha}, f_{\alpha} \in V_x \), we have the following (unique) Wiener-Itô chaos expansion

\[ f = \sum_{\alpha \in J} f_{\alpha} H_{\alpha}, f_{\alpha} \in V_x, \]

with
\[ |f|_{\rho,q,V_x}^2 = \sum_{\alpha \in J} |f_{\alpha}|_{V_x(\alpha!)}^2 (2N^\rho)_{\alpha}, \]
and
\[ \left( \alpha! \right)^{1+\rho} = \prod_{j=1}^{2N^\rho} (\alpha_j!)^{1+\rho}. \]
with

$$||f||^2_{\rho,q} = \sum_{\alpha \in J} f^2_\alpha(\alpha!)^{1+\rho}(2N)^q\alpha.$$  

Note that since $$||f||^2_{0,0} = \sum_{\alpha \in J} f^2_\alpha$$, the stochastic functions in $$S^{\rho,q}$$ become more regular since the Fourier-like coefficients, 

$$|f_\alpha| \leq \frac{||f||_{\rho,q}}{(\alpha!)^{1+\rho}(2N)^q},$$  

decay more rapidly as $$|\alpha| = \sum_j \alpha_j$$ or the length of $$\alpha$$ increases. Hence, carrying out numerical analysis in a given Kondratiev space is equivalent to assuming that the SPDE solution satisfies certain stochastic regularity conditions governed by the decay of its Fourier coefficients (6). The dual of $$S^{\rho,q}$$, referred to as the space of stochastic distributions or generalized random variables, is then identified with $$S^{-\rho,-q}$$ (for further details on stochastic distribution spaces, see, for example [15, 16, 18]). For $$\rho \in [0,1]$$ and $$q \in \mathbb{R}$$, defining the Kondratiev test function spaces

$$S^\rho = \bigcap_{q \geq 0} S^{\rho,q}$$

and the Kondratiev distribution spaces

$$S^{-\rho} = \bigcup_{q \geq 0} S^{-\rho,-q},$$

the following continuous embeddings hold [16, p. 34]

$$S^1 \hookrightarrow S^\rho \hookrightarrow S^0 \hookrightarrow L^2_\mu(S') \hookrightarrow S^{-0} \hookrightarrow S^{-\rho} \hookrightarrow S^{-1}.$$  

Consider $$V'_x$$ the dual space of $$V_x$$ and $$\prec \cdot, \cdot \succ$$ the dual pairing on $$V'_x \times V_x$$. The space $$S^{-\rho,-q,V_x}$$ can be viewed as the dual space of $$S^{\rho,q,V'_x}$$ through the pairing

$$\langle F, f \rangle_{S^{-\rho,-q,V'_x},S^{\rho,q,V_x}} = \sum_{\alpha \in J} \alpha! \cdot f_\alpha \succ \alpha!, \ F \in S^{-\rho,-q,V'_x}, f \in S^{\rho,q,V_x},$$  

which is well defined since

$$|\langle F, f \rangle_{S^{-\rho,-q,V'_x},S^{\rho,q,V_x}}| \leq \sum_{\alpha \in J} \|F_\alpha\|_{\alpha!} \cdot \|f_\alpha\|_{\alpha!} \alpha! (\alpha!)^{1+\rho}(2N)^{-q\alpha} (\alpha!)^{1+\rho}(2N)^{-q\alpha}$$

$$\leq \left( \sum_{\alpha \in J} \|F_\alpha\|_{\alpha!}^2 \alpha! (\alpha!)^{1-\rho}(2N)^{-q\alpha} \right)^{1/2} \left( \sum_{\alpha \in J} \|f_\alpha\|_{\alpha!}^2 \alpha! (\alpha!)^{1+\rho}(2N)^{q\alpha} \right)^{1/2}$$

$$= ||F||_{-\rho,-q,V'_x} ||f||_{\rho,q,V_x} < +\infty$$

using Cauchy-Schwarz’s inequality.

We shall now focus on the stochastic weak formulation associated with the parabolic SPDE model with Dirichlet boundary conditions. The SPDE solution
u is sought in \( V_\chi \otimes V_t \otimes V_\omega \) where \( V_\chi = H_0^1(D) \), \( V_t = L^2(0, T) \) and \( V_\omega \) is a stochastic distribution space \( S^{-\rho,-q} \), with \( q > 0 \). Henceforth, we shall use the notations \( \mathcal{W} := H_0^1(D) \otimes S^{-\rho,-q} \) (\( S_0^{-\rho,-q,1} \) in short) and \( \mathcal{W}' := H^{-1}(D) \otimes S_0^{\rho,q,1} (S_0^{\rho,q,-1}) \) in brief, where \( H^{-1}(D) \) is the dual of \( H_0^1(D) \). We then assume that the SPDE solution is such that \( u \in L^2(0,T;\mathcal{W}) \) and \( \partial u \partial t \in L^2(0,T;\mathcal{W}') \), meaning that

\[
\int_0^T \left( \|u(\cdot, s; \cdot)\|_{\mathcal{W}}^2 + \left\| \frac{\partial u}{\partial t}(\cdot, s; \cdot) \right\|_{\mathcal{W}'}^2 \right) \, ds < +\infty.
\]

The source term is also assumed to be such that \( f \in L^2(0,T;\mathcal{W}') \). When deriving \textit{a priori} error estimates in section 4 we shall require additional regularity assumptions for \( u, f \) and its temporal partial derivatives (see Theorem 4.1 and Remark 1). We shall also specify in section 4 the norm which is used for estimating the solution approximation.

We focus now on the derivation of the weak formulation of the SPDE model (1). By definition since \( \frac{\partial u}{\partial t}(\cdot, t; \cdot) \) and \( f(\cdot, t; \cdot) \in \mathcal{W}' \), the test-functions should belong to \( \mathcal{W} \). However, using the dual pairing \( \langle \cdot, \cdot \rangle_{\mathcal{W}'\times\mathcal{W}} \) in the weak formulation is not easy in practice. Instead, an inner product \( (\cdot, \cdot)_{-\rho,-l,0} \) will be used for some parameter \( l \). In our case we shall consider \( l = q + r \) with \( r > 0 \); the role played by this parameter \( l \) will be explained later in section 4 (see Theorem 4.1).

To proceed further in the derivation of the weak formulation, we borrow ideas from [26] to address embeddings of Kondratiev spaces. Since \( -l < -\rho \), we have \( S_0^{-\rho,-q,1} \hookrightarrow S_0^{-\rho,-l,1} \) from (4). Since \( H_0^1(D) \hookrightarrow L^2(D) \), the embedding \( S_0^{-\rho,-l,1} \hookrightarrow S^{-\rho,-l,0} \) holds (see (5)). Next, consider \( V = S_0^{-\rho,-l,1} \), \( H = S^{-\rho,-l,0} \). From \( V \hookrightarrow H \cong H' \hookrightarrow V' \) where \( H \) is identified with its dual \( H' \) through the Riesz representation theorem, it follows that \( S^{-\rho,-l,0} \hookrightarrow S_{\rho,l}^{-1} \). The last embedding \( S_{\rho,l}^{-1} \hookrightarrow S_{\rho,q}^{-1} \) holds by applying again (4) with \( q < l \). In summary, the following continuous dense embeddings hold

\[
\begin{align*}
S_0^{-\rho,-q,1} & \hookrightarrow S^{-\rho,-l,1} \hookrightarrow S^{-\rho,-l,0} \cong H' \hookrightarrow S_{\rho,l}^{-1} \hookrightarrow S_{\rho,q}^{-1} \\
\mathcal{W} & \hookrightarrow V \hookrightarrow H \hookrightarrow V' \hookrightarrow \mathcal{W}'.
\end{align*}
\]

In such a case it is known that the dual pairing \( \langle \cdot, \cdot \rangle_{\mathcal{W}'\times\mathcal{W}} \) coincides (up to continuous injections)\(^1\) with the inner product \( \langle \cdot, \cdot \rangle_H = (\cdot, \cdot)_{-\rho,-l,0} \), that is,

\[
(h, v)_{-\rho,-l,0} = \langle h, v \rangle_{\mathcal{W}'\times\mathcal{W}}, \quad h \in S^{-\rho,-l,0}, \quad v \in \mathcal{W}.
\]

Since by definition \( \frac{\partial u}{\partial t}, f \in L^2(0,T;\mathcal{W}') \), it follows that

\[
\langle \frac{\partial u}{\partial t} + Lu - f, v \rangle_{\mathcal{W}'\times\mathcal{W}} = \left( \frac{\partial u}{\partial t} + Lu - f, v \right)_{-\rho,-l,0}, \quad \forall v \in \mathcal{W}.
\]

\(^1\)Consider an Hilbert space \( H \) equipped with an inner product \( \langle \cdot, \cdot \rangle_H \) and let \( \mathcal{W} \) be a dense subspace of \( H \). The rigged Hilbert space [27] consists of the following continuous dense embeddings \( \mathcal{W} \hookrightarrow H \hookrightarrow \mathcal{W}' \), where \( \mathcal{W}' \) denotes the dual of \( \mathcal{W} \) and \( H \) is identified with its dual \( H' \). For \( h \in H \) and any \( v \in \mathcal{W} \), Riesz theorem leads to \( (h, Id(v))_H = \langle h', Id(v) \rangle_{H'\times\mathcal{W}} \) for some \( h' \in H' \), where \( Id : \mathcal{W} \rightarrow H \) is a continuous injection (or canonical map). Using the continuous injection \( Id' : H' \rightarrow \mathcal{W}' \) which is defined as the restriction to \( \mathcal{W} \) of any \( h' \in H' \), that is, \( \langle Id'(h'), v \rangle_{\mathcal{W}'\times\mathcal{W}} = \langle h', Id(v) \rangle_{H'\times\mathcal{W}} \) for \( v \in \mathcal{W} \), it follows that \( (h, Id(v))_H = \langle Id'(h'), v \rangle_{\mathcal{W}'\times\mathcal{W}} \) for \( h \in H, v \in \mathcal{W} \) (see [26, Remark 3.2]). For notational convenience, we then use a slight abuse of notation in the previous relation by removing the continuous injections and assimilating \( h \) and \( h' \) to each other since \( H \cong H' \), which leads to (9).
Denoting $A(u, v) := (Lu, v)_{-\rho,-l,0}$ where spatial integration by parts might have been used, the weak formulation of (1) is written as

\[
\left( \frac{\partial u}{\partial t}, v \right)_{-\rho,-l,0} + A(u, v) = (f, v)_{-\rho,-l,0}, \forall v \in \mathcal{S}_0^{\rho,-q,1}, \quad t > 0,
\]

with $u(x, 0; \omega) = u_0(x; \omega) \mu$-a.e. on $D \times S$. For the sake of notational convenience, note that we omit the temporal dependency of $u$ and $f$. From [26, p. 66], there exists a continuous embedding between the space to which those functions $u$ belong and $C^0(0, T; \mathcal{S}^{-\rho,-l,0})$, meaning that it makes sense to consider the trace $u(\cdot, 0; \cdot)$. Note that we require the minimal regularity assumptions for $u$ and $\frac{\partial u}{\partial t}$ in (10). Later in section 4, some additional regularity conditions for $u$, $f$, its temporal partial derivatives and $u_0$ will be needed when deriving error estimates (for a specific example, see Remark 1).

Next, we assume that the bilinear form $A$ is $\alpha_c$-continuous and $\alpha_e$-elliptic with respect to the norm $\| \cdot \|_{-\rho,-l,1}$ (as in [25]):

\[
\exists \alpha_c > 0 \text{ such that } \forall u, v \in \mathcal{S}_0^{\rho,-q,1}, \quad |A(u, v)| \leq \alpha_c \|u\|_{-\rho,-l,1} \|v\|_{-\rho,-l,1}, \quad (11)
\]

\[
\exists \alpha_e > 0 \text{ such that } \forall u \in \mathcal{S}_0^{\rho,-q,1}, \quad A(u, u) \geq \alpha_e \|u\|_{-\rho,-l,1}^2. \quad (12)
\]

In the present analysis, the bilinear form $A$ does not depend explicitly on time but only implicitly through the SPDE solution $u$, namely, $A = A(u(\cdot, t; \cdot), v)$. It makes sense to consider the norm of the SPDE solution $\|u\|_{-\rho,-l,1}$ in (11-12) since $u(\cdot, t; \cdot) \in \mathcal{S}_0^{\rho,-q,1}$ implies $u(\cdot, t; \cdot) \in \mathcal{S}_0^{\rho,-l,1}$ for $q < l$ from (8). From Poincaré inequality, the norms $\| \cdot \|_{H^l(D)}$ and $\| \cdot \|_{H^1(D)}$ are equivalent meaning that $u(\cdot, t; \cdot) \in \mathcal{S}^{\rho,-l,1}$ and $\|u\|_{-\rho,-l,1} < +\infty$.

To guarantee the existence, uniqueness and regularity of the solution of (10), problem dependent assumptions are needed. For example, the existence and uniqueness of the solution for a general class of Wick-type parabolic SPDEs is proved in [26]. Other results can be found in the literature for generalized elliptic SPDE models. Several steady-state stochastic diffusion models have been studied in the white noise setting by considering classical pointwise and Wick products [31]. The existence and uniqueness of the classical SPDE solution can be guaranteed for log-normal random diffusivity fields [13]. For Wick-type diffusion models, the existence and uniqueness of the SPDE solution have been established for more general random diffusivity fields [20].

In Appendix A, the continuity and coercivity conditions (11-12) are proved for a class of second-order SPDEs with deterministic operators $\mathcal{L}$ and random forcing terms $f$. However, it is not clear at this stage how to address the general case of classical-type SPDEs with random parametrized operators; more specifically, what are the regularity conditions the random coefficients need to satisfy such that the assumptions (11-12) hold.

Next, we introduce the finite-dimensional spaces $V^h$ and $(V^M)^{-\rho,-q}$ used for the numerical approximation of (10). First, let $\mathcal{T}$ be a triangulation of $D$ consisting of a finite collection of triangles (resp. tetrahedra) $T_\alpha$ such that $T_\alpha \cap T_\beta = \emptyset$ for $\alpha \neq \beta$, $\bigcup T_\alpha = D$, and such that no vertex lies in the interior of an edge (resp. a face) of another triangle (resp. tetrahedron). We consider a family of triangulations $\mathcal{T}_h$. 
with mesh-size $h$, $h \in [0, 1]$, which are supposed to be non-degenerate, i.e., there exists $\mu > 0$ such that

$$\text{diam}(B_T) \geq \mu \text{diam}(T),$$

for all $T \in \mathcal{T}_h$ and $h \in [0, 1]$, where $B_T$ is the largest ball contained in $T$, and such that

$$\max\{\text{diam}(T), \ T \in \mathcal{T}_h\} \leq h \text{diam}(D).$$

In some cases, we shall consider quasi-uniform triangulations for which

$$\min\{\text{diam}(B_T), \ T \in \mathcal{T}_h\} \geq \mu h \text{diam}(D)$$

holds, for all $h \in [0, 1]$, $\mu > 0$. Note that quasi-uniform triangulations are non-degenerate.

The finite-dimensional subspace $V^h_x \subset H^1_0(D)$ is then defined by

$$V^h_x = \text{span}\{\phi_i(x)\}_{i=1}^n \tag{13}$$

where $\phi_i : \mathcal{T}_h \to \mathbb{R}$ denotes a piecewise linear continuous FE basis function linked to the $i$-th node of $\mathcal{T}_h$ and that vanishes on $\partial D$. Next, to define the finite-dimensional random space, we consider the following set of multi-indices

$$J_{M,p} = \{\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_M), \ \alpha_j \in \mathbb{N}, \ |\alpha| \leq p\}, \tag{14}$$

which is of cardinality $\frac{(M+p)!}{M!p!}$. We can thus define $V^{M,p}_\omega$ as follows:

$$V^{M,p}_\omega = \left\{ f = \sum_{\alpha \in J_{M,p}} f_{\alpha} H_{\alpha}, \ f_{\alpha} \in \mathbb{R} \right\}. \tag{15}$$

Note that $V_x \otimes V^{M,p}_\omega$ is isomorphic to $\text{span}\left(\sum_{\alpha \in J_{M,p}} f_{\alpha} H_{\alpha}, \ f_{\alpha} \in V_x\right)$.

Returning back to the numerical solution of (10), we consider a $\theta$-weighted temporal discretization scheme with $\theta \in [0, 1]$ which results in the following weak problem:

Find $u^m_{h,p,\omega} \in V^h_x \otimes V^{M,p}_\omega$, $0 \leq m \leq N_t$, such that

$$\left\{ \begin{array}{l}
\left( \frac{u^{m+1}_{h,p,\omega} - u^m_{h,p,\omega}}{\Delta t}, v_{h,p,\omega} \right)_{-\rho,-l,0} + A(u^{m+\theta}_{h,p,\omega}, v_{h,p,\omega}) = (f^m_{h,p,\omega}, v_{h,p,\omega})_{-\rho,-l,0}, \\
(u^0_{h,p,\omega} - u_0, v_{h,p,\omega})_{-\rho,-l,0} = 0,
\end{array} \right. \tag{16}$$

for all $v_{h,p,\omega} \in V^h_x \otimes V^{M,p}_\omega$, where $u^m_{h,p,\omega}$ and $f^m_{h,p,\omega}$ denote the approximate solution and source term computed at time $t^m = m\Delta t$, for $0 \leq m \leq N_t$ and $\Delta t = \frac{T}{N_t}$,
respectively. More precisely, \( u_{h,p}^{m} \) is defined as

\[
u_{h,p}^{m} = \sum_{|\alpha| \leq p} \sum_{i=1}^{n} c_{\alpha,i}^{m} \phi_i(x) H_\alpha(\omega)
\]

with coefficients \( c_{\alpha,i}^{m} \) to be determined. The same kind of expansion is used for the approximate source term \( f_{h,p}^{m} \). In (16), \( u_{h,p}^{m+\theta} \) and \( f_{h,p}^{m+\theta} \) are defined as

\[
u_{h,p}^{m+\theta} = \theta u_{h,p}^{m+1} + (1 - \theta) u_{h,p}^{m}, \quad (17)
\]

and

\[
f_{h,p}^{m+\theta} = \theta f_{h,p}^{m+1} + (1 - \theta) f_{h,p}^{m}, \quad (18)
\]

The cases \( \theta = 0, \theta = \frac{1}{2} \) and \( \theta = 1 \) correspond to the forward Euler, Crank-Nicolson and backward Euler schemes, respectively.

3. Stability analysis

In this section, we study the stability of the temporal discretization scheme (16) using ideas from the analysis in [24]. We first show that (16) is unconditionally stable for \( \theta \in \left[\frac{1}{2}, 1\right] \) (see Lemma 3.1) and subsequently we prove that (16) is conditionally stable for \( \theta \in [0, \frac{1}{2}] \) under some restrictions (see Lemma 3.2).

**Lemma 3.1:** Let \( u_{h,p}^{m} \in \mathcal{V}_h \otimes \mathcal{V}_\omega^{M,p,\omega} \) be the solution of (16) with \( \theta \in \left[\frac{1}{2}, 1\right] \), where \( A \) is \( \alpha_c \)-continuous and \( \alpha_e \)-elliptic on \( \mathcal{W} \) with respect to the norm \( \| \cdot \|_{\rho,-l,0} \), \( l = q + r, r > 0 \). If the spatial triangulation is non-degenerate, then the following inequality holds:

\[
\max_{k=1,\ldots,N_t} \| u_{h,p,\omega}^{k} \|_{\rho,-l,0}^2 \leq \| u_{h,p,\omega}^{0} \|_{\rho,-l,0}^2 + \frac{\Delta t}{\alpha_e} \sum_{m=0}^{N_t-1} \| f_{h,p,\omega}^{m+\theta} \|_{\rho,-l,0}^2 \quad (19)
\]

where \( f_{h,p,\omega}^{m+\theta} \) is given by (18).

**Proof:** Writing

\[
\left( \frac{u_{h,p,\omega}^{m+1} - u_{h,p,\omega}^{m}}{\Delta t}, u_{h,p,\omega}^{m+\theta} \right)_{\rho,-l,0} + A(u_{h,p,\omega}^{m+\theta}, \cdot) = (f_{h,p,\omega}^{m+\theta}, \cdot)_{\rho,-l,0},
\]

using the equality

\[
u_{h,p,\omega}^{m+\theta} = \Delta t \left( \theta - \frac{1}{2} \right) \frac{u_{h,p,\omega}^{m+1} - u_{h,p,\omega}^{m}}{\Delta t} + \frac{u_{h,p,\omega}^{m+1} + u_{h,p,\omega}^{m}}{2},
\]
and using the \( \alpha_e \)-ellipticity of \( A \) on \( V_x^{h} \otimes V_\omega^{M,p,w} \subset W \), we have

\[
\Delta t \left( \theta - \frac{1}{2} \right) \left[ \frac{|u_{h,p_w}^{m+1} - u_{h,p_w}^m|}{\Delta t} \right]_{-\rho,-1,0}^2 + \frac{1}{2\Delta t} \left( f_{h,p_w}^{m+\theta} - f_{h,p_w}^m, u_{h,p_w}^{m+1} + u_{h,p_w}^m \right)_{-\rho,-1,0} \]

\[
+ \alpha_e \left[ |u_{h,p_w}^{m+\theta}|_{-\rho,-1,1}^2 \right] \leq \left[ |f_{h,p_w}^{m+\theta}|_{-\rho,-1,0} \right] \left[ |u_{h,p_w}^{m+\theta}|_{-\rho,-1,0} \right] \tag{20}
\]

For every \( w_{h,p,w} \in V_x^{h} \otimes V_\omega^{M,p,w} \), it can be seen that \( \left| w_{h,p,w} \right|_{-\rho,-1,1} \geq \left| w_{h,p,w} \right|_{-\rho,-1,0} \). Indeed, considering \( w_{h,p,w} = \sum_{\alpha \in J_{m,p,w}} w_{\alpha} H_{\alpha} \) with \( w_{\alpha} \in V_x^{h} \), we have

\[
\left| w_{h,p,w} \right|_{-\rho,-1,1}^2 = \sum_{\alpha \in J_{m,p,w}} \left| w_{\alpha} \right|_{L^2(D)}^2 \left( \alpha ! \right)^{1-\theta \left( 2N \right)}^{-l \alpha} \leq \sum_{\alpha \in J_{m,p,w}} \left| w_{\alpha} \right|_{H^1(D)}^2 \left( \alpha ! \right)^{1-\theta \left( 2N \right)}^{-l \alpha} = \left| w_{h,p,w} \right|_{-\rho,-1,1}^2.
\]

We can then apply the previous inequality to \( \left| u_{h,p_w}^{m+\theta} \right|_{-\rho,-1,1}^2 \). Next, applying the inequality \( 2ab \leq a^2 + b^2 \) \((a, b \geq 0, \epsilon > 0)\) with \( \epsilon = \alpha_e > 0 \) to the right-hand side of (20), and using the fact that \( \theta - \frac{1}{2} \geq 0 \), it follows that

\[
\frac{1}{2\Delta t} \left( \left| u_{h,p_w}^{m+1} \right|_{-\rho,-1,0}^2 - \left| u_{h,p_w}^m \right|_{-\rho,-1,0}^2 \right) + \alpha_e \left| u_{h,p_w}^{m+\theta} \right|_{-\rho,-1,0}^2 \leq \frac{1}{2} \frac{\left| f_{h,p_w}^{m+\theta} \right|_{-\rho,-1,0}^2}{\alpha_e} + \alpha_e \left| u_{h,p_w}^{m+\theta} \right|_{-\rho,-1,0}^2.
\]

Reordering terms, we get

\[
\left| u_{h,p_w}^{m+1} \right|_{-\rho,-1,0}^2 - \left| u_{h,p_w}^m \right|_{-\rho,-1,0}^2 + \alpha_e \Delta t \left| u_{h,p_w}^{m+\theta} \right|_{-\rho,-1,0}^2 \leq \frac{\Delta t}{\alpha_e} \left| f_{h,p_w}^{m+\theta} \right|_{-\rho,-1,0}^2.
\]

leading to

\[
\left| u_{h,p_w}^{m+1} \right|_{-\rho,-1,0}^2 \leq \left| u_{h,p_w}^m \right|_{-\rho,-1,0}^2 + \frac{\Delta t}{\alpha_e} \left| f_{h,p_w}^{m+\theta} \right|_{-\rho,-1,0}^2.
\]

We then deduce (19) by induction. \( \square \)

**Lemma 3.2:** Consider the same assumptions as in Lemma 3.1 with a quasi-uniform spatial discretization. For \( \theta \in \left[ 0, \frac{1}{2} \right] \), the following stability condition

\[
\max_{k=1, \ldots, N_t} \left| u_{h,p_w}^k \right|_{-\rho,-1,0}^{2} \leq \left| u_{h,p_w}^0 \right|_{-\rho,-1,0}^{2} + \frac{\Delta t}{\alpha_e} \sum_{m=0}^{N_t-1} \left| f_{h,p_w}^{m+\theta} \right|_{-\rho,-10}^{2} \tag{21}
\]

holds under the time-step restriction

\[
\frac{\Delta t}{h^2} \leq \frac{2\alpha_e(C_D^2 + 1) - 4\epsilon^2 C_D^2}{(C_D^2 + 1)(1 - 2\theta)\alpha_e^2(C_i^2)^2(1 + \epsilon)}, \quad 0 < \epsilon \leq \left( \frac{\alpha_e(C_D^2 + 1)}{C_D^2} \right)^{1/2} \tag{22}
\]
with \( c_t = (1 - 2\theta)(1 + \frac{1}{\epsilon})\Delta t + \frac{1}{4\epsilon^2} \). The constants \( C_t^* \) and \( C_T \) are defined by (25) and (29), respectively.

**Proof:** Using (20), we have

\[
\frac{1}{2\Delta t} \left( \left| \left| v_{h,p}^{m+1} \right| \right|_{-\rho,-l,0}^2 - \left| \left| v_{h,p}^m \right| \right|_{-\rho,-l,0}^2 \right) + \alpha_c \left| \left| u_{h,p}^{m+\theta} \right| \right|_{-\rho,-l,1}^2
\]

\[
\leq \Delta t \left( \frac{1}{2} - \theta \right) \left| \left| \frac{u_{h,p}^{m+1} - u_{h,p}^m}{\Delta t} \right| \right|_{-\rho,-l,0}^2 + \left| \left| f_{h,p}^{m+\theta} \right| \right|_{-\rho,-l,0} \left| \left| u_{h,p}^{m+\theta} \right| \right|_{-\rho,-l,0}. \tag{23}
\]

In order to estimate \( \left| \left| \frac{u_{h,p}^{m+1} - u_{h,p}^m}{\Delta t} \right| \right|_{-\rho,-l,0}^2 \), we substitute \( v_{h,p} = \frac{u_{h,p}^{m+1} - u_{h,p}^m}{\Delta t} \) in (16) and use the inequality (11), which yields

\[
\left| \left| \frac{u_{h,p}^{m+1} - u_{h,p}^m}{\Delta t} \right| \right|_{-\rho,-l,0}^2 \leq \left| \left| f_{h,p}^{m+\theta} \right| \right|_{-\rho,-l,0} \left| \left| \frac{v_{h,p}^{m+1} - v_{h,p}^m}{\Delta t} \right| \right|_{-\rho,-l,0} - A \left( \frac{v_{h,p}^{m+\theta}}{\Delta t}, \frac{u_{h,p}^{m+1} - u_{h,p}^m}{\Delta t} \right) - A \left( \frac{v_{h,p}^{m+\theta}}{\Delta t}, \frac{u_{h,p}^m}{\Delta t} \right) \left| \left| A \left( \frac{v_{h,p}^{m+1} - v_{h,p}^m}{\Delta t}, \frac{u_{h,p}^m}{\Delta t} \right) \right| \right|_{-\rho,-l,0}^2 + \alpha_c \left| \left| v_{h,p}^{m+\theta} \right| \right|_{-\rho,-l,1} \left| \left| \frac{u_{h,p}^{m+1} - u_{h,p}^m}{\Delta t} \right| \right|_{-\rho,-l,0}^2 \cdot \tag{24}
\]

To proceed further, we invoke the following discrete inverse inequality that holds for quasi-uniform meshes (see [10], Theorem 4.5.11):

\[
\left| \left| w_h \right| \right|_{H^1(\Omega)} \leq (C_t^*/h) \left| \left| w_h \right| \right|_{L^2(\Omega)}, \quad \forall w_h \in V_h^h, \tag{25}
\]

where \( C_t^* \) is a constant independent of \( h \). Hence it follows that

\[
\left| \left| w_{h,p} \right| \right|_{-\rho,-l,1} \leq (C_t^*/h) \left| \left| w_{h,p} \right| \right|_{-\rho,-l,0}, \quad \forall w_{h,p} \in V_h^h \otimes V_h^{M,p}. \tag{26}
\]

Applying the previous inequality to \( \frac{u_{h,p}^{m+1} - u_{h,p}^m}{\Delta t} \) in (24), we have

\[
\left| \left| \frac{u_{h,p}^{m+1} - u_{h,p}^m}{\Delta t} \right| \right|_{-\rho,-l,0} \leq \left| \left| f_{h,p}^{m+\theta} \right| \right|_{-\rho,-l,0} + \alpha_c C_t^*/h \left| \left| u_{h,p}^{m+\theta} \right| \right|_{-\rho,-l,1}. \tag{26}
\]

Squaring the previous inequality and applying \( (a + b)^2 \leq (1 + \epsilon)a^2 + (1 + \frac{1}{\epsilon})b^2 \) \((a, b \geq 0, \epsilon > 0)\), we get:

\[
\left| \left| \frac{u_{h,p}^{m+1} - u_{h,p}^m}{\Delta t} \right| \right|_{-\rho,-l,0}^2 \leq (1 + \epsilon) \left( \frac{\alpha_c C_t^*/h}{1 + \frac{1}{\epsilon}} \right) \left| \left| u_{h,p}^{m+\theta} \right| \right|_{-\rho,-l,1}^2 + \left| \left| f_{h,p}^{m+\theta} \right| \right|_{-\rho,-l,0}^2 \cdot \tag{26}
\]
Next, substituting (26) in (23), we obtain
\[
\frac{1}{2\Delta t} \left( ||v_{h,p}^{m+1}||^2_{\rho,-l,0} - ||v_{h,p}^{m}||^2_{\rho,-l,0} \right) + \left( \alpha_e - \Delta t \left( \frac{1}{2} - \theta \right) \left( 1 + \epsilon \right) \left( \frac{\alpha_e C^*_e}{h} \right)^2 \right) ||v_{h,p}^{m+\theta}||^2_{\rho,-l,1} \leq \Delta t \left( \frac{1}{2} - \theta \right) \left( 1 + \frac{1}{\epsilon} \right) ||f_{h,p}^{m+\theta}||^2_{\rho,-l,0} + ||f_{h,p}^{m+\theta}||_{\rho,-l,0} ||u_{h,p}^{m+\theta}||_{\rho,-l,0}. \tag{27}
\]

Using the inequality $2ab \leq \gamma a^2 + \frac{1}{\gamma} b^2$ ($a, b \geq 0, \gamma > 0$) with $\gamma = 4\epsilon^2$, we obtain
\[
||f_{h,p}^{m+\theta}||_{\rho,-l,0} ||u_{h,p}^{m+\theta}||_{\rho,-l,0} \leq 2\epsilon^2 ||u_{h,p}^{m+\theta}||^2_{\rho,-l,0} + \frac{1}{8\epsilon^2} ||f_{h,p}^{m+\theta}||^2_{\rho,-l,0}. \tag{28}
\]

For estimating $||v_{h,p}^{m+\theta}||^2_{\rho,-l,0}$ we use the following Poincaré inequality
\[
||w_h||_{L^2(D)} \leq C_D ||w_h||_{H^1(D)}, \quad \forall w_h \in V_h^D \subset H^1_0(D), \tag{29}
\]
where $C_D$ is a constant which only depends on the spatial dimension $d$ and the diameter of $D$. Using the preceding inequality, we have
\[
||w_h||^2_{L^2(D)} \leq \frac{C_D^2}{C_D^2 + 1} ||w_h||^2_{H^1(D)}, \quad \forall w_h \in V_h^D \subset H^1_0(D),
\]
and
\[
||w_{h,p}||^2_{\rho,-l,0} \leq \frac{C_D^2}{C_D^2 + 1} ||w_{h,p}||^2_{\rho,-l,1}, \quad \forall w_{h,p} \in V_h^M \subset V_h^M \subset V \otimes S^{-\rho,-q}.
\]
Hence we obtain the inequality
\[
||f_{h,p}^{m+\theta}||_{\rho,-l,0} ||u_{h,p}^{m+\theta}||_{\rho,-l,0} \leq 2\epsilon^2 \frac{C_D^2}{C_D^2 + 1} ||u_{h,p}^{m+\theta}||^2_{\rho,-l,1} + \frac{1}{8\epsilon^2} ||f_{h,p}^{m+\theta}||^2_{\rho,-l,0}.
\]
Substituting the preceding inequality in (27), we have
\[
\frac{1}{2\Delta t} \left( ||v_{h,p}^{m+1}||^2_{\rho,-l,0} - ||v_{h,p}^{m}||^2_{\rho,-l,0} \right) + \left( \alpha_e - \Delta t \left( 1 - 2\theta \right) \left( 1 + \epsilon \right) \frac{\alpha_e C^*_e}{2h^2} \right) \left( \frac{C_D^2}{C_D^2 + 1} \right) ||u_{h,p}^{m+\theta}||^2_{\rho,-l,1} \leq \Delta t \frac{1}{2} \left( 1 - 2\theta \right) \left( 1 + \frac{1}{\epsilon} \right) ||f_{h,p}^{m+\theta}||^2_{\rho,-l,0} + \frac{1}{8\epsilon^2} ||f_{h,p}^{m+\theta}||^2_{\rho,-l,0}. \tag{30}
\]
To ensure stability in (30), we require the sufficient condition $\alpha_e - \beta \Delta t - \lambda \geq 0$ with $\beta = \frac{(1-2\theta)(1+\epsilon)\alpha_e C^*_e}{2h^2} > 0$ and $\lambda = \frac{2\epsilon^2 C_D^2}{C_D^2 + 1} > 0$. Next, assuming that $\alpha_e - \lambda \geq 0$,
that is,

$$0 < \epsilon \leq \left( \frac{\alpha e (C_D^2 + 1)}{2C_D^2} \right)^{1/2},$$  \hspace{1cm} (31)$$

it follows that

$$0 \leq \Delta t \leq \frac{\alpha - \lambda}{\beta} = \frac{2h^2\alpha e (C_D^2 + 1) - 4\epsilon^2 C_D^2}{(C_D^2 + 1)(1 - 2\theta)\alpha^2 (C_i^2) (1 + \epsilon)}.$$  \hspace{1cm} (32)$$

Assuming that (31-32) hold, we finally get

$$||u_{h,p,\omega}^m||_{-\rho,-l,0}^2 \leq ||u_{h,p,\omega}^m||_{-\rho,-l,0}^2 + \Delta t \left( (1 - 2\theta) \left( 1 + \frac{1}{\epsilon} \right) \Delta t + \frac{1}{4\epsilon^2} \right) ||f_{h,p,\omega}^{m+\theta}||_{-\rho,-l,0}^2$$ \hspace{1cm} (33)$$

from which we deduce the expected result (21) by induction. \hspace{1cm} \square$$

4. \textit{A priori} error estimation

Our aim is to estimate an upper bound for the error metric

$$\max_{m=1,\ldots,M} ||e_{h,p,\omega}^m||_{-\rho,-l,0}, \text{ where } e_{h,p,\omega}^m = u(\cdot,t^m;\cdot) - u_{h,p,\omega}^m$$ \hspace{1cm} (34)$$

with \( l = q + r \) and \( r > 0 \). For the purpose of this analysis, additional spatial and stochastic regularity assumptions will be made for \( u, f \) and its temporal partial derivatives, and \( u_0 \). We shall now prove \textit{a priori} error estimates for two different cases, corresponding to \( \theta \in \left[ \frac{1}{2}, 1 \right] \) and \( \theta \in [0, \frac{1}{2}] \).

\textbf{Theorem 4.1:} Consider \( \rho \in [0, 1] \), \( l = q + r \) with \( r > 0 \). Suppose there exists a weak solution of (1) satisfying (10) with \( u \in L^2(0,T;S_0^{-\rho,-q,1} \cap S^{-\rho,-q,2}), \frac{\partial u}{\partial t} \in L^2(0,T;S^{\rho,q,-1}), f \in L^2(0,T;S^{\rho,q,-1}) \). In addition, assume that \( f \) and its temporal partial derivatives are smooth enough and that \( u_0 \) belongs at least to \( S^{-\rho,-q,2} \), ensuring that \( \frac{\partial u}{\partial t} \in L^2(0,T;S^{-\rho,-q,2}) \) and \( \frac{\partial^2 u}{\partial t^2} \in L^2(0,T;S^{-\rho,-q,0}) \). Assume that the bilinear form \( A \) is \( \alpha_c \)-continuous and \( \alpha_e \)-elliptic with respect to the norm \( ||\cdot||_{-\rho,-l,1} \).

Let \( u_{h,p,\omega}^m \in V_h \otimes V_{\omega}^{M,p,\omega} \) denote the solution of the \( \theta \)-scheme (16) with \( \theta \in \left[ \frac{1}{2}, 1 \right] \), where the finite-dimensional spaces \( V_h \) and \( V_{\omega}^{M,p,\omega} \) are given by (13) and (15), respectively. The spatial triangulation is supposed to be non-degenerate. Then the following error estimate holds

$$\max_{m=1,2,\ldots,M} ||e_{h,p,\omega}^m||_{-\rho,-l,0} \leq C_1 h + C_2 \Delta t + C_3 c(M,p_\omega,r),$$ \hspace{1cm} (35)$$
where the constants $C_1, C_2, C_3, C^* > 0$ are independent of $h$ and $\Delta t$, and only depends on the analytical solution as follows:

$$C_1 = \frac{C^*\alpha_e}{\alpha_e} \left( \max_{m=1,2,\ldots,N_t} \| u(\cdot, t^m, \cdot) \|_{-\rho,-t,2} + \| u_0 \|_{-\rho,-t,2} + \frac{2}{\sqrt{\alpha_e}} K_1(u, T) \right),$$

$$C_2 = \sqrt{\frac{2}{\alpha_e}} K_2(u, T),$$

$$C_3 = \frac{\alpha_e}{\alpha_e} \left( \max_{m=1,2,\ldots,N_t} \| u(\cdot, t^m, \cdot) \|_{-\rho,q,1} + \| u_0 \|_{-\rho,q,1} + \frac{2}{\sqrt{\alpha_e}} K_3(u, T) \right),$$

with

$$K_1(u, T) = \left( \int_0^T \left( \left\| \frac{\partial u}{\partial t} \right\|_{-\rho,-t,2} \right)^2 ds \right)^{1/2},$$

$$K_2(u, T) = \left( \int_0^T \left( \left\| \frac{\partial^2 u}{\partial t^2} \right\|_{-\rho,-t,0} \right)^2 ds \right)^{1/2},$$

$$K_3(u, T) = \left( \int_0^T \left( \left\| \frac{\partial u}{\partial t} \right\|_{-\rho,-t,1} \right)^2 ds \right)^{1/2}.$$ 

where $\left( \frac{\partial u}{\partial t} \right)_s, \left( \frac{\partial^2 u}{\partial t^2} \right)_s$ denote $\frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial t^2}$ evaluated at time $s$. The constant $c(M, p_\omega, r)$ given by (39) is such that $c(M, p_\omega, r) \to 0$ when $M \to +\infty$ and $p_\omega \to +\infty$.

**Proof:** First, we split the error as follows:

$$e_{h,p_\omega}^m = u(\cdot, t^m ; \cdot) - u_{h,p_\omega}^m = u(\cdot, t^m ; \cdot) - Pu(\cdot, t^m ; \cdot) + Pu(\cdot, t^m ; \cdot) - u_{h,p_\omega}^m \quad (36)$$

where $P : \mathcal{W} \to \mathcal{V}_x^h \otimes \mathcal{V}_{\omega}^{M,p_\omega}$ is the elliptic projection defined by the Galerkin conditions:

$$A(Pu(\cdot, t ; \cdot), v_{h,p_\omega}) = A(u(\cdot, t ; \cdot), v_{h,p_\omega}), \quad \forall v_{h,p_\omega} \in \mathcal{V}_x^h \otimes \mathcal{V}_{\omega}^{M,p_\omega}.$$ 

The existence and uniqueness of $Pu(\cdot, t ; \cdot)$ is a consequence of the Lax-Milgram theorem. Hence we need to estimate

$$\max_{m=1,\ldots,N_t} \| e_{h,p_\omega}^m \|_{-\rho,-t,0} \leq \max_{m=1,\ldots,N_t} \| u_{h,p_\omega}^m \|_{-\rho,-t,0} + \max_{m=1,\ldots,N_t} \| e_{h,p_\omega}^m \|_{-\rho,-t,0}. \quad (37)$$

Since we have

$$A(\eta_{h,p_\omega}^m, v_{h,p_\omega}) = 0, \quad \forall v_{h,p_\omega} \in \mathcal{V}_x^h \otimes \mathcal{V}_{\omega}^{M,p_\omega},$$

we can use the following error estimate given in [13, Remark 4.7] along with Céa’s
lemma: For \( l = q + r \) with \( r > 0 \), we have, for \( m = 1, \ldots, N_t \),

\[
\begin{align*}
||\eta_m^{h,p,\omega}||_{-\rho,-t,0} & \leq ||\eta_m^{h,p,\omega}||_{-\rho,-t,1} \\
& \leq \frac{\alpha_e}{\alpha_e} \left( C^* h ||u(\cdot, t^m ; \cdot)||_{-\rho,-t,2} + c(M, p_\omega, r) ||u(\cdot, t^m ; \cdot)||_{-\rho,-q,1} \right) \\
\end{align*}
\]  

(38)

with

\[
c(M, p_\omega, r) = \frac{1}{2^r} \max \left\{ \frac{1}{2^r M}, \frac{1}{(p_\omega + 1)^r} \right\}
\]

(39)

and \( C^* \) is a constant independent of \( h \) and \( \Delta t \). It is to be noted that the stochastic error due to the Kondratiev space truncation provided by Galvis and Sarkis [13] is sharper than the estimations given by Benth and Gjerde [9, Theorem 5.1] and by Cao [11, Theorem 2].

We focus now on estimating an upper bound for the term \( ||\xi_m^{h,p,\omega}||_{-\rho,-t,0} \). From (36) and (17) we substitute

\[
u_m^{h,p,\omega} = u(\cdot, t^m ; \cdot) - (\eta_m^{h,p,\omega} + \xi_m^{h,p,\omega})
\]

and

\[
u_m^{h,p,\omega} + \theta = \theta (u(\cdot, t^{m+1} ; \cdot) - (\eta_m^{h,p,\omega} + \xi_m^{h,p,\omega})) + (1 - \theta)(u(\cdot, t^m ; \cdot) - (\eta_m^{h,p,\omega} + \xi_m^{h,p,\omega}))
\]

into the \( \theta \)-scheme (16). Hence, using (18), we note that \( \xi_m^{h,p,\omega} \) are the solution of

\[
\left( \frac{\xi_m^{m+1} - \xi_m^{h,p,\omega}}{\Delta t}, v_{h,p,\omega} \right)_{-\rho,-t,0} + A(\xi_m^{m+\theta}, v_{h,p,\omega}) = (\varphi_m^{m+\theta}, v_{h,p,\omega})_{-\rho,-t,0}
\]

(40)

with

\[
\varphi_m^{m+\theta} = u(\cdot, t^{m+1} ; \cdot) - u(\cdot, t^m ; \cdot) - \left( \frac{\partial u}{\partial t} \right)_{t=m+\theta} - \frac{\eta_m^{h,p,\omega} - \eta_m^{h,p,\omega}}{\Delta t}
\]

and

\[
\left( \frac{\partial u}{\partial t} \right)_{t=m+\theta} = \theta \left( \frac{\partial u}{\partial t} \right)_{t=m+1} + (1 - \theta) \left( \frac{\partial u}{\partial t} \right)_{t=m}.
\]

We then apply the stability result (19) to (40) which yields

\[
\max_{m=1,\ldots,N_t} ||\xi_m^{h,p,\omega}||_{-\rho,-t,0} \leq ||\xi_0^{h,p,\omega}||_{-\rho,-t,0} + \frac{\Delta t}{\alpha_e} \sum_{m=0}^{N_t-1} ||\varphi_m^{m+\theta}||_{-\rho,-t,0}^2,
\]

(41)
where
\[
\|\varphi_{m+\theta} - \varphi_{m} - \theta_{m+\theta} - \theta_{m}\|_{-\rho,-l,0} \leq \left| \frac{u(\cdot, t^{m+1}; \cdot) - u(\cdot, t^{m}; \cdot)}{\Delta t} - \frac{\partial u}{\partial t}(\cdot, t^{m+1}; \cdot) \right|_{-\rho,-l,0}^{(I)} + \left| \frac{\eta_{m+1} - \eta_{m}}{\Delta t} \right|_{-\rho,-l,0}^{(II)}.
\]

We first consider the estimation of the term (I). Noting that
\[
\frac{u(\cdot, t^{m+1}; \cdot) - u(\cdot, t^{m}; \cdot)}{\Delta t} = \theta \left( \frac{u(\cdot, t^{m+1}; \cdot) - u(\cdot, t^{m}; \cdot)}{\Delta t} - \left( \frac{\partial u}{\partial t} \right)_{t^{m+1}} \right) + (1 - \theta) \left( \frac{u(\cdot, t^{m+1}; \cdot) - u(\cdot, t^{m}; \cdot)}{\Delta t} - \left( \frac{\partial u}{\partial t} \right)_{t^{m}} \right)
\]
and using Taylor’s formulas with integral remainder
\[
u(\cdot, t^{m+1}; \cdot) = u(\cdot, t^{m}; \cdot) + \Delta t \left( \frac{\partial u}{\partial t} \right)_{t^{m}} + \int_{t^{m}}^{t^{m+1}} (t^{m+1} - s) \left( \frac{\partial^2 u}{\partial t^2} \right)_s ds
\]
\[
u(\cdot, t^{m}; \cdot) = u(\cdot, t^{m+1}; \cdot) - \Delta t \left( \frac{\partial u}{\partial t} \right)_{t^{m+1}} + \int_{t^{m+1}}^{t} (t^{m} - s) \left( \frac{\partial^2 u}{\partial t^2} \right)_s ds,
\]
we obtain:
\[
(I) \leq \frac{\theta}{\Delta t} \left| \int_{t^{m}}^{t^{m+1}} (t^{m} - s) \left( \frac{\partial^2 u}{\partial t^2} \right)_s ds \right|_{-\rho,-l,0} + \frac{1 - \theta}{\Delta t} \left| \int_{t^{m}}^{t^{m+1}} (t^{m+1} - s) \left( \frac{\partial^2 u}{\partial t^2} \right)_s ds \right|_{-\rho,-l,0}.
\]

Applying a Minkowsky inequality, we get
\[
(I) \leq \frac{\theta}{\Delta t} \int_{t^{m}}^{t^{m+1}} |t^{m} - s| \left| \left( \frac{\partial^2 u}{\partial t^2} \right)_s \right|_{-\rho,-l,0} ds + \frac{1 - \theta}{\Delta t} \int_{t^{m}}^{t^{m+1}} |t^{m+1} - s| \left| \left( \frac{\partial^2 u}{\partial t^2} \right)_s \right|_{-\rho,-l,0} ds.
\]

Since \( |t^{m} - s| \leq \Delta t \) and \( |t^{m+1} - s| \leq \Delta t \), it follows that
\[
(I) \leq \int_{t^{m}}^{t^{m+1}} \left| \left( \frac{\partial^2 u}{\partial t^2} \right)_s \right|_{-\rho,-l,0} ds \leq \sqrt{\Delta t} \left( \int_{t^{m}}^{t^{m+1}} \left| \left( \frac{\partial^2 u}{\partial t^2} \right)_s \right|_{-\rho,-l,0}^2 ds \right)^{1/2}.
\]

(43)
using a Cauchy-Schwarz’s inequality. Expanding the analytical solution as $u = \sum_{\alpha \in J} u_\alpha(x, t) H_\alpha$, we get

$$\int_{t^m}^{t^{m+1}} \left( \left\| \frac{\partial^2 u}{\partial t^2} (\cdot, s) \right\|_{-\rho, -l, 0}^2 \right) ds = \sum_{\alpha \in J} \int_{t^m}^{t^{m+1}} \left( \left\| \frac{\partial^2 u_\alpha}{\partial t^2} (\cdot, s) \right\|_{L^2(D)}^2 \right) (\alpha!)^{1-\rho}(2N)^{-l\alpha}$$

$$= \sum_{\alpha \in J} \int_{D} \left( \int_{t^m}^{t^{m+1}} \left\| \frac{\partial^2 u_\alpha(x, s)}{\partial t^2} \right\|_{(y_m(u(x)))^2}^2 ds \right) dx (\alpha!)^{1-\rho}(2N)^{-l\alpha}$$

$$= \left\| y^m(u) \right\|_{-\rho, -l, 0}^2$$

(44)

with $y^m(u) = \sum_{\alpha \in J} y^m_\alpha(u) H_\alpha$. Hence, from (43) and (44), we deduce that

$$(I) \leq \sqrt{\Delta t} \left\| y^m(u) \right\|_{-\rho, -l, 0}$$

(45)

with $y^m(u) = \sum_{\alpha \in J} y^m_\alpha(u) H_\alpha$.

Concerning the second term $(II)$, we use the fact that

$$A \left( \eta_{h,p\omega}^{m+1} - \eta_{h,p\omega}^m, \eta_{h,p\omega} \right) = 0, \forall \eta_{h,p\omega} \in V_h^x \otimes V_{\omega}^{M,p\omega}.$$

Using the error estimate (38) again leads to

$$(II) \leq \frac{\alpha_c}{\alpha_e} \left( C^* h \left\| \frac{u(\cdot, t^{m+1}; \cdot) - u(\cdot, t^m; \cdot)}{\Delta t} \right\|_{-\rho, -l, 2} \right. \right. \right.$$

$$+ \left. \left. e(M, p\omega, r) \left\| \frac{u(\cdot, t^{m+1}; \cdot) - u(\cdot, t^m; \cdot)}{\Delta t} \right\|_{-\rho, -q, 1} \right) \right).$$

(46)

We have

$$\left\| \frac{u(\cdot, t^{m+1}; \cdot) - u(\cdot, t^m; \cdot)}{\Delta t} \right\|_{-\rho, -q, 1} = \frac{1}{\Delta t} \left\| \int_{t^m}^{t^{m+1}} \left( \frac{\partial u}{\partial t} \right) ds \right\|_{-\rho, -q, 1}$$

$$= \frac{1}{\Delta t} \left( \sum_{\alpha \in J} \left\| \tilde{w}_\alpha^m \right\|_{H^1(D)}^2 (\alpha!)^{1-\rho}(2N)^{-q\alpha} \right)^{1/2}$$

(47)

with

$$\tilde{w}_\alpha^m(x) = \int_{t^m}^{t^{m+1}} \frac{\partial u_\alpha}{\partial t}(x, s) ds.$$
Applying Minkowsky and Cauchy-Schwarz’s inequalities lead to

\[
\|\bar{w}_m^{\alpha}\|_{H^1(\mathcal{D})} \leq \sqrt{\Delta t} \left( \int_{t_m}^{t_{m+1}} \left\| \frac{\partial u_{\alpha}}{\partial t}(\cdot, s) \right\|_{H^1(\mathcal{D})}^2 \, ds \right)^{1/2}
\]

\[
= \sqrt{\Delta t} \left( \int_{\mathcal{D}} \left( \int_{t_m}^{t_{m+1}} \sum_{0 \leq |\beta| \leq 1} \left| D_x^\beta \left( \frac{\partial u_{\alpha}}{\partial t}(x, s) \right) \right|^2 \, ds \right)^{1/2} \right)^{1/2}
\]

\[
= \sqrt{\Delta t} \| z_m^{\alpha}(u) \|_{L^2(\mathcal{D})}. \quad (49)
\]

Note that \( D_x^\beta \) is defined by \( D_x^\beta = \frac{\partial (\beta)}{\partial \alpha x_1 \cdots \partial \alpha x_d} \), for every multi-indices \( \beta = (\beta_1, \ldots, \beta_d) \). Combining (47) and (49), we have

\[
\left| \frac{u(\cdot, t_{m+1}; \cdot) - u(\cdot, t_m; \cdot)}{\Delta t} \right|_{-p, q, 1} \leq \frac{1}{\sqrt{\Delta t}} \| z^m(u) \|_{-p, q, 0} \quad (50)
\]

with \( z^m(u) = \sum_{\alpha \in J} z_{\alpha}^m(u) H_{\alpha} \). Similarly, we obtain the following inequality

\[
\left| \frac{u(\cdot, t_{m+1}; \cdot) - u(\cdot, t_m; \cdot)}{\Delta t} \right|_{-p, l, 2} \leq \frac{1}{\sqrt{\Delta t}} \| \zeta^m(u) \|_{-p, l, 0} \quad (51)
\]

with \( \zeta^m(u) = \sum_{\alpha \in J} \zeta_{\alpha}^m(u) H_{\alpha} \) and

\[
\zeta_{\alpha}^m(u)(x) = \left( \int_{t_m}^{t_{m+1}} \sum_{0 \leq |\beta| \leq 2} \left| D_x^\beta \left( \frac{\partial u_{\alpha}}{\partial t}(x, s) \right) \right|^2 \, ds \right)^{1/2}. \quad (52)
\]

From (46), (50) and (51) it follows that

\[
(II) \leq \frac{1}{\sqrt{\Delta t}} \frac{\alpha c}{\alpha e} \left( C^e h \| \zeta^m(u) \|_{-p, l, 0} + c(M, p_\omega, r) \| z^m(u) \|_{-p, q, 0} \right) \quad (53)
\]

To complete the estimation in (41), we invoke the Galerkin orthogonality condition corresponding to the initial value, which gives

\[
(u_{h, p_\omega}^0 - u_0, v_{h, p_\omega})_{-p, l, 0}, \quad \forall v_{h, p_\omega} \in V_h^b \otimes V_{\omega}^{M, p_\omega}. \quad (54)
\]

Noting that \( u_{h, p_\omega}^0 - u_0 = -(\xi_{h, p_\omega}^0 + \zeta_{h, p_\omega}^0) \), and taking \( v_{h, p_\omega} = \xi_{h, p_\omega}^0 \) in (54), we get

\[
\| \xi_{h, p_\omega}^0 \|_{-p, l, 0} \leq \| \eta_{h, p_\omega}^0 \|_{-p, l, 0} \leq \frac{\alpha c}{\alpha e} \left( C^e h \| u_0 \|_{-p, l, 2} + c(M, p_\omega, r) \| u_0 \|_{-p, q, 1} \right) \quad (55)
\]

Gathering equations (37), (38), (41), (42), (45), (53), (55) and reordering terms,
we obtain
\[
\max_{m=1,2,...,N_t} \left| e^{m}_{h,p_{c}} \right| \leq C^* \frac{\alpha e}{\alpha e} \left[ \max_{m=1,2,...,N_t} \left| u(\cdot,t^m,\cdot) \right|^{\rho_{-t,2}} + \| u_0 \|^{\rho_{-t,2}} + \frac{2}{\alpha e} \left( \sum_{m=0}^{N_t-1} \| \zeta^m(u) \|^{2_{\rho_{-t,0}}} \right)^{1/2} \right] + \sqrt{\frac{2}{\alpha e} \Delta t} \left( \sum_{m=0}^{N_t-1} \| y^m(u) \|^{2_{\rho_{-t,0}}} \right)^{1/2} + \frac{\alpha e}{\alpha e} c(M,p_\omega,r) \left[ \max_{m=1,2,...,N_t} \left| u(\cdot,t^m,\cdot) \right|^{\rho_{-q,1} + \| u_0 \|^{\rho_{-q,1}}} + \frac{2}{\alpha e} \left( \sum_{m=0}^{N_t-1} \| z^m(u) \|^{2_{\rho_{-t,0}}} \right)^{1/2} \right]. \tag{56}
\]

In the preceding inequality, we obtain constants which are dependent of \( \Delta t \) through the summations over \( N_t \) terms since \( N_t = T/\Delta t \). In order to obtain constants independent of \( \Delta t \) in the upper bound, we use the following relation
\[
(K_2(u,T))^2 = \int_0^T \left\| \left( \frac{\partial^2 u}{\partial t^2} \right)_s \right\|^{2_{\rho_{-t,0}}} ds = \sum_{m=0}^{N_t-1} \sum_{\alpha \in J} \left( \alpha! \right)^{1-\rho} (2N)^{-l \alpha} \int_{t_m}^{t_{m+1}} \left\| \frac{\partial^2 u_\alpha}{\partial t^2} (\cdot,s) \right\|_{L^2(D)}^{2_{\rho_{-t,0}}} ds \]
\[
= \sum_{m=0}^{N_t-1} \sum_{\alpha \in J} \left( \alpha! \right)^{1-\rho} (2N)^{-l \alpha} \| y^m_\alpha(u) \|_{L^2(D)}^{2_{\rho_{-t,0}}} \]
\[
= \sum_{m=0}^{N_t-1} \| y^m(u) \|_{L^2(D)}^{2_{\rho_{-t,0}}}. \tag{57}
\]

Similarly, it can be seen that
\[
(K_1(u,T))^2 = \int_0^T \left\| \left( \frac{\partial u}{\partial t} \right)_s \right\|^{2_{\rho_{-t,2}}} ds = \sum_{m=0}^{N_t-1} \| \zeta^m(u) \|_{L^2(D)}^{2_{\rho_{-t,0}}} \tag{58}
\]
and
\[
(K_3(u,T))^2 = \int_0^T \left\| \left( \frac{\partial u}{\partial t} \right)_s \right\|^{2_{\rho_{-q,1}}} ds = \sum_{m=0}^{N_t-1} \| z^m(u) \|_{L^2(D)}^{2_{\rho_{-t,0}}}. \tag{59}
\]

The combination of (56), (57), (58) and (59) finally yields the expected error estimate (35), which concludes the proof. \( \square \)

**Remark 1:** The regularity assumptions required in the statement of Theorem 4.1 ensure that the constants \( C_1, C_2 \) and \( C_3 \) which are involved in the error estimate (35) are finite. For \( q < l \), the embedding \( S^{-\rho_{-q,2}} \hookrightarrow S^{-\rho_{-t,2}} \) ensures that \( \| u(\cdot,t^m,\cdot) \|_{L^2(D)} \leq \| u^m(\cdot,\cdot) \|_{L^2(D)} \) is finite in \( C_1 \), since by definition \( u \in L^2(0,T;S^{-\rho_{-q,2}}) \). Obviously since \( S^{-\rho_{-q,2}} \hookrightarrow S^{-\rho_{-q,1}} \), we have \( \| u(\cdot,t^m,\cdot) \|_{L^2(D)} \leq \| u^m(\cdot,\cdot) \|_{L^2(D)} \) is finite in \( C_3 \).
Since the initial value function is such that \( u_0 \in S^{-\rho,-q,2}, \|u_0\|_{-\rho,-l,2} < +\infty \) in \( C_1 \) and \( \|u_0\|_{-\rho,-q,1} < +\infty \) in \( C_3 \), using the same arguments. Next, assuming that \( f \), its temporal partial derivatives and \( u_0 \) are smooth enough to ensure that \( \frac{\partial u_0}{\partial t} \in L^2(0,T;S^{-\rho,-q,2}) \) and \( \frac{\partial^2 u_0}{\partial x^2} \in L^2(0,T;S^{-\rho,-q,0}) \), it follows that \( K_1(u,T), K_2(u,T) \) and \( K_3(u,T) \) are finite. It is worth noting that such regularity conditions are problem dependent.

To illustrate, consider the stochastic heat equation \( \frac{\partial u}{\partial t} - \Delta u = f \) with random source term \( f \), random initial value function \( u_0 \) and homogeneous Dirichlet boundary conditions, which belongs to the class of second-order SPDEs described in Appendix A. Using parabolic regularity arguments for spatial operators [28] \((\partial \mathcal{D})\) (\( \partial \mathcal{D} \) is supposed to be smooth here), it can be shown that under data regularity assumptions \( f \in L^2(0,T;S^{-\rho,-q,2}), \frac{\partial f}{\partial t} \in L^2(0,T;S^{-\rho,-q,0}) \) and \( u_0 \in S^{-\rho,-q,3} \), there exists a unique solution that satisfies the following conditions \( u \in L^2(0,T;S^{-\rho,-q,4}), \frac{\partial u}{\partial t} \in L^2(0,T;S^{-\rho,-q,2}) \) and \( \frac{\partial^2 u}{\partial x^2} \in L^2(0,T;S^{-\rho,-q,0}) \), implying that \( C_1, C_2 \) and \( C_3 \) are well-defined.

**Theorem 4.2:** Under the same assumptions as in Theorem 4.1 with a quasi-uniform spatial discretization, consider the \( \theta \)-scheme (16) with \( \theta \in [0, \frac{1}{2}] \). Assuming the following restriction on the time-step

\[
\frac{\Delta t}{h^2} \leq \frac{2\alpha_e(C_D^2 + 1) - 4\epsilon^2 C_D^2}{(C_D^2 + 1)(1 - 2\theta)\alpha_e^2(C_i^*)^2(1 + \epsilon)} , \quad 0 < \epsilon \leq \left( \frac{\alpha_e(C_D^2 + 1)}{2C_D^2} \right)^{1/2},
\]

the error estimation (35) holds with

\[
C_1 = \frac{C_i^*}{\alpha_e} \left( \max_{m=1,2,...,N_t} \|u(\cdot, t^m, \cdot)\|_{-\rho,-l,2} + \|u_0\|_{-\rho,-l,2} + 2\sqrt{c_e} K_1(u,T) \right),
\]

\[
C_2 = \sqrt{2c_e} K_2(u,T),
\]

\[
C_3 = \frac{\alpha_e}{\alpha_e} \left( \max_{m=1,2,...,N_t} \|u(\cdot, t^m, \cdot)\|_{-\rho,-q,1} + \|u_0\|_{-\rho,-q,1} + 2\sqrt{c_e} K_3(u,T) \right),
\]

and \( \alpha_e = (1 - 2\theta)(1 + \frac{1}{2})\Delta t + \frac{1}{4\epsilon^2} \). The constants \( C_i^* \) and \( C_D \) are defined by (25) and (29), respectively.

**Proof:** This result is obtained by combining the arguments used in the proof of Theorem 4.1 along with the conditional stability result given by Lemma 3.2. \( \square \)

**Remark 2:** Consider the case when the SPDE coefficients are modeled as generalized random fields which are discretized using the Karhunen-Loève (KL) expansion scheme [19]. In such problems one would expect that if the KL eigenvalues decay sufficiently fast then the approximation error should go to zero when the number of terms retained in the KL expansion \( (M) \) is increased. It is to be noted that (35) provides an useful error estimate since \( c(M, p_\omega, r) \rightarrow 0 \) for \( M \rightarrow +\infty \) and \( p_\omega \rightarrow +\infty \). However, further work is needed to study the stochastic regularity of the solution of specific SPDE models. This will provide insights into appropriate values of the stochastic regularity parameters \( \rho \) and \( q \), which can be subsequently used in computational implementations of stochastic finite element methods. It is worth noting that some information about \( \rho \) and \( q \) is available when considering Wick-type SPDEs. In [25, 26], the existence of a (unique) solution is proved for
a class of elliptic and parabolic Wick-SPDEs, for $\rho = -1$ and $q$ small enough, exploiting some properties of the Wick product. In [26], convergence rates are also provided in the parabolic case for $\rho = -1$ and $q < -\ast$.

**Remark 3:** From (35), it can be seen that the error scales linearly in $h$ and $\Delta t$ when using linear piecewise finite element basis functions. This behavior which is weaker than known results for classical PDEs corresponds to the convergence rate obtained for Wick-type parabolic SPDEs [26]. When considering spatial domain with smooth boundary $\partial D$, the spatial convergence rate in (38) can be improved using higher order finite element approximations. Consider $V_\mathbf{x} = H^1_0(D) \cap H^{k+1}(D)$ with $k \geq 1$ and let $V_h$ be the subspace spanned by piecewise polynomials of degree at most $k$ that vanish on $\partial D$. Then, instead of (38), the following estimate holds:

$$
\| \eta_{h,p\omega}^m \|_{-\rho,-t,1} \leq \frac{\alpha_c}{\alpha_e} \left( \tilde{C}^* h^k \| u(\cdot, t^m, \cdot) \|_{-\rho, -t, k+1} + c(M, p, \omega, r) \| u(\cdot, t^m, \cdot) \|_{-\rho, -q, 1} \right),
$$

where $\tilde{C}^*$ is a constant independent of $h$.

The error estimate in Theorem 4.1 thus becomes

$$
\max_{m=1,2,...,N_t} \| e_{h,p\omega}^m \|_{-\rho,-t,0} \leq C_1 h^k + C_2 \Delta t + C_3 c(M, p, \omega, r),
$$

where $C_1$ is now defined as

$$
C_1 = \frac{\tilde{C}^* \alpha_c}{\alpha_e} \left( \max_{m=1,2,...,N_t} \| u(\cdot, t^m, \cdot) \|_{-\rho, -t, k+1} + \| u_0 \|_{-\rho, -t, k+1} + \frac{2}{\alpha_e} \tilde{K}_1(u, T) \right)
$$

with

$$
\tilde{K}_1(u, T) = \left( \int_0^T \left\| \frac{\partial u}{\partial t}(\cdot, s, \cdot) \right\|_{-\rho, -t, k+1}^2 ds \right)^{1/2}
$$

**Remark 4:** Sharper error estimates can be obtained when considering suitable elliptic regularity conditions for the adjoint problem. Using duality arguments, it can be shown (see Appendix B) that the error in (38) scales as $O(h^2 + (c(M, p, \omega, r))^2)$ when considering linear piecewise finite element basis functions. As a consequence, the estimate in (35) converges at the rate $O(h^2 + \Delta t + (c(M, p, \omega, r))^2)$.

**Acknowledgments**

This research is funded by an NSERC Discovery Grant and the Canada Research Chairs program.

**References**


Appendix A. Continuity and ellipticity conditions for a class of SPDEs

We consider a class of SPDE models with second-order deterministic operators, homogenous Dirichlet boundary conditions and a random forcing term $f(x, t; \omega)$. The spatial domain is supposed to be bounded. The second-order deterministic
operators considered here are of the form

\[ \mathcal{L} = - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial}{\partial x_j} \right) + \sum_{i=1}^{d} b_i(x) \frac{\partial}{\partial x_i} + c(x) \]

with the following conditions

\[ a_{ij} \in C^1(\bar{D}), \ i, j = 1, \ldots, d, \]

\[ \exists \bar{c} > 0 \text{ such that } \sum_{i,j=1}^{d} a_{ij}(x)w_iw_j \geq \bar{c} \sum_{i=1}^{d} w_i^2 \text{ for all } w = (w_1, \ldots, w_d) \in \mathbb{R}^d, \]

\[ b_i \in C^0(\bar{D}), \ i = 1, \ldots, d, \]

\[ c \in C^0(\bar{D}), \]

\[ c(x) - \frac{1}{2} \sum_{i=1}^{d} \frac{\partial}{\partial x_i} b_i(x) \geq 0. \]

Define \( \bar{a} = \max_{i,j=1,\ldots,d} \max_{x \in \bar{D}} |a_{ij}(x)|, \) \( \bar{b} = \max_{i=1,\ldots,d} \max_{x \in \bar{D}} |b_i(x)| \) and \( \bar{c} = \max_{x \in \bar{D}} |c(x)|. \) After using a spatial integration by parts, the bilinear form is given by

\[ A(u, v) = (\mathcal{L} u, v)_{-\rho, -l, 0} \]

\[ = \sum_{\alpha \in J} \left[ \sum_{i,j=1}^{d} \left( a_{ij} \frac{\partial u_\alpha}{\partial x_j}, \frac{\partial v_\alpha}{\partial x_i} \right) L^2(\bar{D}) + \sum_{i=1}^{d} \left( b_i \frac{\partial u_\alpha}{\partial x_i}, v_\alpha \right) L^2(\bar{D}) \right. \]

\[ + \left. (c u_\alpha, v_\alpha)_{L^2(\bar{D})} \right] (\alpha!)^{1-\rho} (2N)^{-l} \alpha. \]

Let us focus on the continuity of \( A \) with respect to \( || \cdot ||_{-\rho, -l, 1} \). First, we have

\[ \sum_{\alpha \in J} (c u_\alpha, v_\alpha)_{L^2(\bar{D})} (\alpha!)^{1-\rho} (2N)^{-l} \alpha \leq \bar{c} (u, v)_{-\rho, -l, 0} \leq \bar{c} ||u||_{-\rho, -l, 1} ||v||_{-\rho, -l, 1}. \quad (A1) \]

Next, using Cauchy-Schwarz’s inequalities we have

\[ \sum_{\alpha \in J} \sum_{i=1}^{d} \left( b_i \frac{\partial u_\alpha}{\partial x_i}, v_\alpha \right) L^2(\bar{D}) (\alpha!)^{1-\rho} (2N)^{-l} \alpha \]

\[ \leq \bar{b} \sum_{i=1}^{d} \left( \frac{\partial u}{\partial x_i}, v \right)_{-\rho, -l, 0} \leq \bar{b} \sum_{i=1}^{d} \left\| \frac{\partial u}{\partial x_i} \right\|_{-\rho, -l, 0} ||v||_{-\rho, -l, 0} \]

\[ \leq \bar{b} \sqrt{d} \left( \sum_{i=1}^{d} \left\| \frac{\partial u}{\partial x_i} \right\|_{-\rho, -l, 0}^2 \right)^{1/2} ||v||_{-\rho, -l, 0} = \bar{b} \sqrt{d} \left\| \nabla u \right\|_{-\rho, -l, 0} ||v||_{-\rho, -l, 0} \]

\[ \leq \bar{b} \sqrt{d} ||u||_{-\rho, -l, 1} ||v||_{-\rho, -l, 1}. \quad (A2) \]
For the bilinear term we have

\[
\left| \sum_{\alpha \in J} \sum_{i,j=1}^{d} \left( a_{ij} \frac{\partial u_{\alpha}}{\partial x_j} \frac{\partial u_{\alpha}}{\partial x_i} \right) \right|_{L^2(D)} \leq \bar{a} \sum_{i,j=1}^{d} \left( \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} \right)_{-\rho,-t,0} \\
\leq \bar{a} \sum_{i=1}^{d} \left( \frac{\partial u}{\partial x_j} \right)_{-\rho,-t,0} \sum_{j=1}^{d} \left( \frac{\partial v}{\partial x_i} \right)_{-\rho,-t,0} \\
\leq \bar{a} d \|\nabla u\|_{-\rho,-t,0} \|\nabla v\|_{-\rho,-t,0} \leq \bar{a} d \|u\|_{-\rho,-t,1} \|\nabla v\|_{-\rho,-t,1}.
\]

(A3)

Combining (A1), (A2) and (A3), it follows that \( A \) is \( \alpha_{c} \)-continuous with respect to the norm \( \|\cdot\|_{-\rho,-t,1} \) with \( \alpha_{c} = \tilde{c} + b \sqrt{d} + \bar{a} d \). Let us focus now on the ellipticity condition. Consider \( A(u, u) = A_1(u, u) + A_2(u, u) \) with

\[
A_1(u, u) = \sum_{\alpha \in J} \sum_{i=1}^{d} \left( a_{ij} \frac{\partial u_{\alpha}}{\partial x_j} \frac{\partial u_{\alpha}}{\partial x_i} \right) \left( \alpha! \right)^{-\rho}(2N)^{-1} \alpha,
\]

\[
A_2(u, u) = \sum_{\alpha \in J} \left[ \sum_{i=1}^{d} \left( b_i \frac{\partial u_{\alpha}}{\partial x_i} u_{\alpha} \right) \right] + \left( cu_{\alpha}, u_{\alpha} \right)_{L^2(D)} \left( \alpha! \right)^{-\rho}(2N)^{-1} \alpha.
\]

From \( \sum_{i,j=1}^{d} \left( a_{ij} \frac{\partial u_{\alpha}}{\partial x_j} \frac{\partial u_{\alpha}}{\partial x_i} \right) \geq \tilde{c} \sum_{i=1}^{d} \left( \frac{\partial u_{\alpha}}{\partial x_i} \right)^2 \), it follows that

\[
A_1(u, u) \geq \tilde{c} \sum_{\alpha \in J} \sum_{i=1}^{d} \left( \frac{\partial u_{\alpha}}{\partial x_i} \right)^2 \left( \alpha! \right)^{-\rho}(2N)^{-1} \alpha
\]

\[
= \tilde{c} d \sum_{i=1}^{d} \left( \frac{\partial u}{\partial x_i} \right)^2 \geq \tilde{c} \|\nabla u\|^2_{-\rho,-t,0} \geq \frac{\tilde{c}}{1 + C_D^2} \|u\|^2_{-\rho,-t,1}
\]

where \( C_D \) denotes Poincaré constant (see (29)). For the second term, reordering terms and using an integration by part, we obtain

\[
A_2(u, u) = \sum_{\alpha \in J} \int_D \left( cu_{\alpha}^2 + \frac{1}{2} \sum_{i=1}^{d} b_i \frac{\partial u_{\alpha}^2}{\partial x_i} \right) dx \left( \alpha! \right)^{-\rho}(2N)^{-1} \alpha
\]

\[
= \sum_{\alpha \in J} \int_D \left( c - \frac{1}{2} \sum_{i=1}^{d} \frac{\partial b_i}{\partial x_i} u_{\alpha}^2 \right) dx \left( \alpha! \right)^{-\rho}(2N)^{-1} \alpha \geq 0,
\]

meaning that \( A \) is \( \alpha_{c} \)-elliptic with respect to the norm \( \|\cdot\|_{-\rho,-1,1} \) with \( \alpha_{c} = \frac{\tilde{c}}{1 + C_D^2} \).
Appendix B. Duality technique for elliptic SPDEs

Consider the weak formulation associated with an elliptic SPDE problem with Dirichlet boundary conditions, that is,

Find \( u \in \mathcal{W} \) such that:

\[
A(u, v) = (f, v)_{-\rho, -l, 0}, \quad \forall v \in \mathcal{W} = (H^1_0(D) \cap H^2(D)) \otimes S^{-\rho, -q},
\]

(B1)

with \( q > 0 \), \( l = q + r \), \( r > 0 \). The bilinear form \( A \) is \( \alpha_e \)-continuous and \( \alpha_e \)-elliptic with respect to the norm \( \| \cdot \|_{-\rho, -l, 1} \).

The adjoint (or dual) problem is given by:

Find \( w \in \mathcal{W} \) such that:

\[
A(v, w) = (g, v)_{-\rho, -l, 0}, \quad \forall v \in \mathcal{W},
\]

(B2)

for which we assume the following elliptic regularity estimates

\[
\|w\|_{-\rho, -l, 2} \leq C_r \|g\|_{-\rho, -l, 0}
\]

\[
\|w\|_{-\rho, -q, 1} \leq \tilde{C}_r \|g\|_{-\rho, -l, 0}
\]

(B3)

hold for all \( g \in \mathcal{W} \). Note that considering the norm \( \| \cdot \|_{-\rho, -l, 2} \) in the first regularity assumption makes sense since \( \|w\|_{-\rho, -l, 2} \leq \|w\|_{-\rho, -q, 2} < +\infty \) for any \( l > q \) and \( w \in \mathcal{W} \).

**Proposition B.1:** Consider \( u \in \mathcal{W} \) the solution of (B1) and let \( u_{h,p} \) be the approximate solution in \( \mathcal{V}_s \otimes \mathcal{V}_{\omega}^{M,p_{\omega}} \), where \( \mathcal{V}_s \) and \( \mathcal{V}_{\omega}^{M,p_{\omega}} \) are given by (13) and (15), respectively. Then the following error estimate holds

\[
\|u - u_{h,p}\|_{-\rho, -l, 0} \leq \frac{\alpha_c}{\alpha_e} \left( h^2 \left( A_1 \|u\|_{-\rho, -l, 2} + A_2 \|u\|_{-\rho, -q, 1} \right) + (c(M, p_{\omega}, r))^2 (B_1 \|u\|_{-\rho, -l, 2} + B_2 \|u\|_{-\rho, -q, 1}) \right). \]

(B4)

**Proof:** Let \( w \) be the solution of (B2) with \( g = u - u_{h,p} \). We have

\[
\|u - u_{h,p}\|_{-\rho, -l, 0} = \|u - u_{h,p}, u - u_{h,p}\|_{-\rho, -l, 0} = A(u - u_{h,p}, w) = A(u - u_{h,p}, w - w_{h,p})
\]

since \( A(u - u_{h,p}, w_{h,p}) = 0 \), for all \( w_{h,p} \in \mathcal{V}_s \otimes \mathcal{V}_{\omega}^{M,p_{\omega}} \). Hence it follows that

\[
\|u - u_{h,p}\|_{-\rho, -l, 0} \leq \alpha_c \|u - u_{h,p}\|_{-\rho, -l, 1} \|w - w_{h,p}\|_{-\rho, -l, 1}.
\]

(B5)

We have the following error estimate from [13]

\[
\|u - u_{h,p}\|_{-\rho, -l, 1} \leq \frac{\alpha_c}{\alpha_e} (C^*h \|u\|_{-\rho, -l, 2} + c(M, p_{\omega}, r) \|u\|_{-\rho, -q, 1}) \]

(B6)
where $c(M, p, r)$ is given by (39). Using the regularity estimates (B3), we have the following error estimate

$$\| w - w_{h,p} \|_{-\rho, -l,1} \leq C^* h \| w \|_{-\rho, -l,2} + c(M, p, r) \| w \|_{-\rho, -q,1}$$

$$\leq \left( C^* C_r h + c(M, p, r) \tilde{C}_r \right) \| u - u_h \|_{-\rho, -l,0}. \quad (B7)$$

Combining (B5), (B6) and (B7), we get

$$\| u - u_{h,p} \|_{-\rho, -l,0} \leq \frac{\alpha_e^2}{\alpha_e} (C^* h \| u \|_{-\rho, -l,2} + c(M, p, r) \| u \|_{-\rho, -q,1})$$

$$\times \left( C^* C_r h + c(M, p, r) \tilde{C}_r \right).$$

Using the inequality $hc(M, p, r) \leq \frac{1}{2} \left( h^2 + (c(M, p, r))^2 \right)$ and reordering terms, we finally get the expected error estimate (B4) with

$$A_1 = C^* 2 C_r + \frac{C^* \tilde{C}_r}{2},$$
$$A_2 = \frac{C^* \tilde{C}_r}{2},$$
$$B_1 = \frac{C^* \tilde{C}_r}{2},$$
$$B_2 = \tilde{C}_r + \frac{C^* \tilde{C}_r}{2}.$$

This concludes the proof. \qed