Block multiplication of matrices

Consider two matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{p \times n}$ and denote the product by $C = AB \in \mathbb{R}^{m \times p}$. The block multiplication of $A$ by $B$ is based on the classical matrix multiplication and uses the fact that $A$ by $B$ are partitioned in blocks. To illustrate how it works, consider the following partitions of $A$ and $B$:

$$
A = \begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix}
$$

with $A_{11} \in \mathbb{R}^{n_1 \times n_2}$, $A_{12} \in \mathbb{R}^{n_1 \times n_2}$, $A_{21} \in \mathbb{R}^{m_1 \times n_2}$, and $A_{22} \in \mathbb{R}^{m_2 \times n_2}$, and

$$
B = \begin{pmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{pmatrix}
$$

with $B_{11} \in \mathbb{R}^{n_1 \times p_1}$, $B_{12} \in \mathbb{R}^{n_1 \times p_2}$, $B_{21} \in \mathbb{R}^{m_2 \times p_1}$, $B_{22} \in \mathbb{R}^{m_2 \times p_2}$, where $m = m_1 + m_2$, $n = n_1 + n_2$ and $p = p_1 + p_2$. Note that the partition of the columns of $A$ must be compatible with the partition of the rows of $B$. Then it can be shown that

$$
C = \begin{pmatrix} C_{11} & C_{12} \\
C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\
A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix}
$$

with $C_{11} \in \mathbb{R}^{m_1 \times p_1}$, $C_{12} \in \mathbb{R}^{m_1 \times p_2}$, $C_{21} \in \mathbb{R}^{m_2 \times p_1}$, $C_{22} \in \mathbb{R}^{m_2 \times p_2}$.

For example, let us prove that $C_{11} = A_{11}B_{11}$. Consider the corresponding element matrix $c_{ij}$ of $C$ with $i \in \{1, 2, \ldots, m_1\}$ and $j \in \{1, 2, \ldots, p_1\}$. By definition of the classical matrix multiplication,

$$
c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}
$$

which can be split as

$$
c_{ij} = \sum_{k=1}^{n_1} a_{ik}b_{kj} + \sum_{k=n_1+1}^{n} a_{ik}b_{kj}.
$$

The preceding summations can be rewritten in terms of the block matrix entries as (be careful with the numbering of indices in the second sum)

$$
c_{ij} = \sum_{k=1}^{n_1} (A_{11})_{ik}(B_{11})_{kj} + \sum_{k=n_1+1}^{n_1+n_2} (A_{12})_{i,k-n_1}(B_{21})_{k-n_1,j}.
$$

Using the change of indices $k' = k - n_1$ in the second sum it follows that

$$
c_{ij} = \sum_{k=1}^{n_1} (A_{11})_{ik}(B_{11})_{kj} + \sum_{k'=1}^{n_2} (A_{12})_{ik'}(B_{21})_{k'j}, \ i \in \{1, 2, \ldots, m_1\}, \ j \in \{1, 2, \ldots, p_1\},
$$

which is nothing but

$$
C_{11} = A_{11}B_{11} + A_{12}B_{21}.
$$

The other blocks $C_{12}, C_{21}$ and $C_{22}$ can be obtained using the same arguments.

Summary: If $A$ and $B$ are partitioned with matrix blocks (of course, more blocks can be used in the partitions that are presented here), the product $C = AB$ can be written in terms of blocks such as $C_{ij} = \sum_k A_{ik}B_{kj}$, i.e., we apply a similar formula used for the classical matrix product except that we are using blocks instead of matrix entries.