Exercises (week 9)

Question 1

Let $V, W, Z$ be vector spaces over the same field, and let $T : V \to W$ and $S : W \to Z$ be linear transformations. Let $S \circ T$ denote the composition of $S$ and $T$, that is, $S \circ T$ is the function $V \to Z$ given by $(S \circ T)(v) = S(T(v))$.

1. Prove that $S \circ T$ is a linear transformation.
2. Let $E, F, G$ be bases for $V, W, Z$ respectively. Prove that $[S \circ T]_{G}^{E} = [S]_{G}^{F} [T]_{E}^{F}$.
3. Let $T : V \to W$ be an invertible linear transformation, and consider the linear transformation $S = T^{-1} : W \to V$. Deduce from (2) that $[T^{-1}]_{E}^{F} = ([T]_{E}^{F})^{-1}$.

Question 2

Given that $S = \{ p \in \mathbb{P}_{4} \text{ such that } p(1) = p(-1) = 0 \}$ is a subspace of $\mathbb{P}_{4}$, find a basis for $S$ and determine the dimension of $S$.

Question 3

Let $V = \text{span}\{\sin(x), \cos(x)\}$. Consider the mapping $T : V \to V$ defined by $T(f(x)) = f'(x)$.

1. Show that $T$ is a linear transformation.
2. Show that $T$ is invertible. What is $T^{-1}$?
3. Compute the matrix representation $[T]_{E}^{E}$ in the basis $E = \{\sin(x), \cos(x)\}$.
4. Compute $[T]_{F}^{F}$ in the basis $F = \{\sin(x) + \cos(x), \sin(x) - \cos(x)\}$.

Question 4

Consider the linear transformation from $\mathbb{P}_{3}$ to $\mathbb{R}^{2}$ defined by $T(a + bx + cx^{2} + dx^{3}) = \left(\begin{array}{c} a + b \\ a - 2c \end{array}\right)$.

1. Prove that $T$ is invertible. Hint: You can show that $T$ is bijective.
2. What is $T^{-1}$?

Question 5

Let $U, V, W$ be vector spaces, and let $T : U \to V$ and $S : V \to W$ be invertible linear transformations. Prove that $S \circ T : U \to W$ is invertible and that $(S \circ T)^{-1} = T^{-1} \circ S^{-1}$. 