Exercises (week 8)

Question 1 Consider the $\mathbb{R}$-vector space $V = (0, +\infty)$ equipped with the following addition and scalar multiplication (see chapter IV):

\[ v + w := vw, \quad \forall v, w \in V \]
\[ \lambda \cdot v := v^\lambda, \quad \forall v \in V, \quad \lambda \in \mathbb{R} \]

(a) Show that $B = \{v_1\}$ with $v_1 > 0, v_1 \neq 1$ is a basis for $V$.
(b) Compute explicitly the coordinate of the zero vector of $V$ in the basis $B$.

Question 2 True/False: The following functions are linear transformations

(a) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (4x + 2y, 0, x + 3z - 1)$
(b) $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x, y, z) = (2x + z, -5z)$
(c) $T : P_n \to \mathbb{R}$ defined by $T(p) = \int_0^1 (x - 2)p(x) + 3xp'(x) \, dx$
(d) $T : C([0, 1]) \to C([0, 1])$ defined by $T(f)(x) = f(x) \sin(x)$
(e) $T : C([0, 1]) \to C([0, 1])$ defined by $T(f)(x) = f(x) + \sin(x)$

Question 3 Let $p_1(x) = \frac{1}{2}x(x - 1), p_2(x) = -x^2 + 1$ and $p_3(x) = \frac{1}{2}x(x + 1)$.

(a) Show that $E = \{p_1, p_2, p_3\}$ is a basis for $P_2$.
(b) Find the coordinates of $q(x) = 2x^2 - 3x + 7$ in the basis $E$. Hint: consider $q(-1), q(0)$ and $q(1)$.

Question 4 Consider $P_1$ with the bases $E = \{1 + x, 2 + 3x\}$ and $F = \{1, x\}$.

(a) Compute the change of basis matrix $P_{E \to F}$ and deduce $P_{F \to E}$.
(b) What are the coordinates of the polynomial $p(x) = -2 + 5x$ in the bases $E$ and $F$?

Question 5 Let $U = \text{span}\{T_0, T_2, T_4\} \subset P_4$, where $T_0, T_2, T_4$ are Chebyshev polynomials

\[ T_0(x) = 1, \quad T_2(x) = -1 + 2x^2, \quad T_4(x) = 1 - 8x^2 + 8x^4. \]

(a) Show that $T = \{T_0, T_2, T_4\}$ is a basis for $U$. Hint: Find the coordinates of $T_0, T_2, T_4$ in a basis for $P_4$, and then use a technique that involves taking the RREF of a certain matrix.
(b) Consider another basis $H = \{H_0, H_2, H_4\}$ for $U$, where $\{H_0, H_2, H_4\}$ are Hermite polynomials

\[ H_0(x) = 1, \quad H_2(x) = -2 + 4x^2, \quad H_4(x) = 12 - 48x^2 + 16x^4. \]

Compute the change of basis matrices $P_{T \to H}$ and $P_{H \to T}$.
(c) Consider the polynomial $p(x) = 3 - 3x^2 + 2x^4$. Compute the coordinates of $p$ with respect to the basis $T$. Deduce its coordinates with respect to the basis $H$.

Question 6 Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ such that

\[ T(1, 0, 1) = (-1, 1, 0, 2), \quad T(0, 1, 1) = (0, 6, -2, 0) \quad \text{and} \quad T(-1, 1, 1) = (4, 2, 1, 0). \]

(a) Show that $F = \{(1, 0, 0), (0, 1, 1), (-1, 1, 1)\}$ is a basis for $\mathbb{R}^3$.
(b) Determine the change of basis matrix $P_{E \to F}$, where $E = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is the standard basis for $\mathbb{R}^3$.
(c) Compute the coordinates of $u = (1, 2, -1)$ in the basis $F$ and deduce $T(u)$.

Question 7 Let $T : V \to W$ be a linear transformation, where $V, W$ are vector spaces. Let $U$ be any subspace of $W$. Define

\[ T^{-1}(U) := \{v \in V \text{ such that } T(v) \in U\}. \]

Show that $T^{-1}(U)$ is a subspace of $V$. 