Exercises (week 5)

Question 1 Let $S = \{1 + x, 1 + kx + x^2, 1 + 2x^2\}$. Determine for which values of $k$ is $\dim(\text{span}(S)) = 3$.

Question 2 Consider two subspaces $U$ and $W$ of a vector space. If $\dim(U) = \dim(W) = 2$, $U \neq W$ and $U \cap W \neq \{0_V\}$, show that $\dim(U \cap W) = 1$.

Question 3 If $\{u, v, w\}$ is a basis for a vector space $V$, determine which of the following are also bases

- $\{u + v, v + w, w + u\}$ Note: there are two different ways of approaching this problem – either showing that the set is linearly independent, or showing that it spans $V$ (because the set contains $\dim(V)$ elements, showing either of these two is sufficient). Both ways are possible using techniques you already know, so it would be educational to try both!
- $\{u, u + v + w\}$

Question 4 Let $V$ be a vector space, let $\{v_1, \ldots, v_k\} \subseteq V$, and let $v \in V$. Let $U = \text{span}\{v_1, \ldots, v_k\}$, and $W = \text{span}\{v_1, \ldots, v_k, v\}$. Show that either $\dim(W) = \dim(U)$, or that $\dim(W) = \dim(U) + 1$. Hint: split the proof into two cases – either $v \in \text{span}\{v_1, \ldots, v_k\}$, or $v \notin \text{span}\{v_1, \ldots, v_k\}$.