Exercises (week 13)

Question 1

(a) Let $\lambda_1, \ldots, \lambda_r$ be the eigenvalues of a matrix $A \in \mathbb{R}^n$, $1 \leq r \leq n$. Show that $A$ is diagonalizable if and only if,

$$\text{algebraic multiplicity of } \lambda_i = n - \text{rank}(A - \lambda_i I) \text{ for each } i$$

(b) Let $A \in \mathbb{R}^{50 \times 50}$ have characteristic polynomial $(\lambda - 12)^30(\lambda - 5)^15\lambda^5$. Fill in the blanks:

$A$ is diagonalizable $\iff$ rank$(A - 12I) =$ ______ and rank$(A - 5I) =$ ______ and rank$(A) =$ ______

Question 2

Let $A$ and $P$ be the matrices

$$A = \begin{pmatrix} -5 & 4 & -4 \\ -4 & 3 & -2 \\ 4 & -4 & 5 \end{pmatrix} \quad P = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix}$$

Use the fact that

$$P^{-1}AP = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

to write down the general solution to $\dot{x} = Ax$.

Question 3 Three tanks of liquid are interconnected such that the volume of liquid in each tank varies as

$$\begin{align*}
\dot{x}_1 &= -2x_1 + x_3 \\
\dot{x}_2 &= -2x_2 + x_3 \\
\dot{x}_3 &= x_1 - 2x_3
\end{align*}$$

where $x_k(t)$ is the volume of liquid in the $k$-th tank.

(a) Write the differential equations above in the form $\dot{x} = Ax$.

(b) Find the eigenvalues of $A$. Is $A$ diagonalizable?

(c) Find a basis for each eigenspace of $A$.

(d) Determine the general solution of $\dot{x} = Ax$.

(e) Find the solution of $\dot{x} = Ax$ with initial conditions $x_1(0) = 100$, $x_2(0) = 0$ and $x_3(0) = 0$.

Question 4 Consider $A, B \in \mathbb{R}^n$.

(a) Show that if there exists a basis for $\mathbb{R}^n$ which consists of eigenvectors of both $A$ and $B$, then $AB$ is diagonalizable and $AB = BA$.

(b) Find an example of two matrices $A, B \in \mathbb{R}^2$ such that $AB = BA$ but which are not both diagonalizable.