So far in class, we have a fun relationship with the definition of a vector space we learned in January: it seems we’re overwhelmingly studying the vector space \( \mathbb{R}^n \), and it’s only occasionally that we study examples of vector spaces besides \( \mathbb{R}^n \). In this chapter, we’ll learn the reason for this: whenever we have a basis for a vector space \( V \), we have a way of “converting” vectors in \( V \) into column matrices of numbers. This means that many techniques we’ve learned to study \( \mathbb{R}^n \) can also be used for other vector spaces \( V \) after we “convert” from \( V \) to \( \mathbb{R}^n \) (or, more generally, \( \mathbb{F}^n \)).

### Day 1: Converting vectors to column matrices

**Definition:** Let \( V \) be a vector space with a basis \( B = \{ v_1, \ldots, v_n \} \). For every vector \( v \in V \) there is a unique collection of scalars \( c_i \) for which \( v = c_1 v_1 + \cdots + c_n v_n \). The coordinates of \( v \) in the basis \( B \) is the column matrix

\[
[v]_B = \begin{pmatrix}
c_1 \\
c_2 \\
\vdots \\
c_n
\end{pmatrix}
\]

We’re using the notation \([v]_B\) for this column matrix because we’re converting \( v \) into a matrix (hence putting brackets around it), and using \( B \) to do so (hence the subscript \( B \)).

**Examples:** In the vector space \( \mathbb{P}_2 \),

- The coordinates of \( 6x^2 + 3x + 5 \) in the basis \( \{ 1, x, x^2 \} \) is \((5\ 3\ 6)^T\).
- The coordinates of \( 6x^2 + 3x + 5 \) in the basis \( \{ 1, x^2, x \} \) is \((5\ 6\ 3)^T\).
- The coordinates of \( 6x^2 + 3x + 5 \) in the basis \( \{ 1 + x + x^2, x, x^2 \} \) is \((5\ -2\ 1)^T\).

Notice that the coordinates of a vector depends on the basis that you choose – even the order that you list the basis elements! Also notice that we can also go backwards: convert a column matrix of numbers into a vector, then back into a column matrix of numbers, you’ll end with the same column matrix you started with. Conversely, if you start with a column matrix of numbers and convert it into a vector, then back into a column matrix of numbers, you’ll end with the same column matrix you started with. This correspondence between these vector spaces is very well-behaved.

**Remark:** Let \( V \) be a vector space, with scalar field \( \mathbb{F} \) and basis \( B = \{ v_1, \ldots, v_n \} \). There is a correspondence

\[
V \leftrightarrow \mathbb{F}^n \\
v \mapsto [v]_B \\
c_1 v_1 + \cdots + c_n v_n \mapsto (c_1, \ldots, c_n)^T
\]

The two functions described in the remark above (one going \( V \to \mathbb{F}^n \) and one going \( \mathbb{F}^n \to V \)) are inverse to each other. This means that if you convert a vector into a column matrix, then convert that back into a vector, you’ll end with the same vector you started with. Conversely, if you start with a column matrix of numbers and convert it into a vector, then back into a column matrix of numbers, you’ll end with the same column matrix you started with. This correspondence between these vector spaces is very well-behaved.

**Theorem** Let \( V \) be a vector space, with field of scalars \( \mathbb{F} \), and basis \( B = \{ v_1, \ldots, v_n \} \).

1. \([0]_B = (0, 0, \ldots, 0)^T\)
2. \([v+w]_B = [v]_B + [w]_B\) for all \( v, w \in V \).
3. \([\lambda \cdot v]_B = \lambda \cdot [v]_B\) for all \( v \in V \) and \( \lambda \in \mathbb{F} \).
4. Let \( \{ w_1, \ldots, w_k \} \subseteq V \). Then \( \{ w_1, \ldots, w_k \} \) is linearly independent if and only if \( \{ [w_1]_B, \ldots, [w_k]_B \} \subseteq \mathbb{F}^n \) is linearly independent.

**Proof**

1. The fact that \([0]_B = (0, 0, \ldots, 0)^T\) follows from the definition of \([0]_B\) and that \( 0 = 0v_1 + \cdots + 0v_n \).
2. Let \( v = c_1 v_1 + \cdots + c_n v_n \) and \( w = d_1 v_1 + \cdots + d_n v_n \). Then

\[
[v + w]_B = [(c_1 + d_1) v_1 + \cdots + (c_n + d_n) v_n]_B = \begin{pmatrix} c_1 + d_1 \\ \vdots \\ c_n + d_n \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} + \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} = [v]_B + [w]_B
\]

3. Let \( v = c_1 v_1 + \cdots + c_n v_n \). Then

\[
[\lambda \cdot v]_B = [\lambda \cdot (c_1 v_1 + \cdots + c_n v_n)]_B = [\lambda c_1 v_1 + \cdots + \lambda c_n v_n]_B = \begin{pmatrix} \lambda c_1 \\ \vdots \\ \lambda c_n \end{pmatrix} = \lambda \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \lambda \cdot [v]_B
\]

4. We will show that \( \{w_1, \ldots, w_k\} \) is linearly dependent if and only if \( \{[w_1]_B, \ldots, [w_k]_B\} \) is linearly dependent\(^1\). We prove each direction of the implication separately.

\( \Rightarrow \) Let \( \{w_1, \ldots, w_k\} \subseteq V \) be linearly dependent. Let \( c_1, \ldots, c_k \in \mathbb{F} \) be a collection of scalars, not all zero, for which \( c_1 w_1 + \cdots + c_k w_k = 0 \). Then

\[
[c_1 w_1 + \cdots + c_k w_k]_B = (0, 0, \ldots, 0)^T
\]

\[
c_1 [w_1]_B + \cdots + c_k [w_k]_B = (0, 0, \ldots, 0)^T
\]

Because \( c_1, \ldots, c_k \) is a collection of scalars, not all zero, for which \( c_1 [w_1]_B + \cdots + c_k [w_k]_B \) is the zero vector in \( \mathbb{F}^n \), it follows that \( \{[w_1]_B, \ldots, [w_k]_B\} \) is linearly dependent.

\( \Leftarrow \) Let \( \{[w_1]_B, \ldots, [w_k]_B\} \subseteq \mathbb{F}^n \) be linearly dependent. Let \( c_1, \ldots, c_k \in \mathbb{F} \) be a collection of scalars, not all zero, for which \( c_1 [w_1]_B + \cdots + c_k [w_k]_B = 0 \). Then

\[
c_1 [w_1]_B + \cdots + c_k [w_k]_B = (0, 0, \ldots, 0)^T
\]

\[
c_1 [w_1]_B + \cdots + c_k [w_k]_B = (0, 0, \ldots, 0)^T
\]

so \( c_1 w_1 + \cdots + c_k w_k \) is the zero vector in \( V \). Therefore, \( \{w_1, \ldots, w_k\} \subseteq V \) is linearly dependent.

Let’s see how we can use the idea of “coordinates” to answer questions about vector spaces besides \( \mathbb{F}^n \).

**Question** Let \( V = \mathbb{R}^2 \), and consider the subspace

\[
\text{span}\left\{ v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, v_3 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, v_4 = \begin{pmatrix} 7 \\ 8 \end{pmatrix} \right\}
\]

Find a basis for this subspace.

**Answer** Consider the following basis for \( V \) (this is not a basis for the subspace! Just for \( V \))

\[
B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}
\]

Then,

\[
[v_1]_B = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad [v_2]_B = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad [v_3]_B = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}, \quad [v_4]_B = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}
\]

Now we can solve our original question by finding a basis for

\[
\text{span}\{[v_1]_B, [v_2]_B, [v_3]_B, [v_4]_B\} \subseteq \mathbb{R}^4
\]

and converting the basis elements (in \( \mathbb{R}^4 \)) back into vectors in \( V = \mathbb{R}^2 \) using the basis \( B \). There are two methods we know of for finding a basis for the span above.

\(^1\)this is equivalent to showing that \( \{w_1, \ldots, w_k\} \) is linearly indep if and only if \( \{[w_1]_B, \ldots, [w_k]_B\} \) is linearly indep
1. Find a basis for the column space of
\[
\begin{pmatrix}
1 & 1 & 3 & 7 \\
1 & 2 & 4 & 8 \\
1 & 1 & 5 & 9 \\
1 & 2 & 6 & 10
\end{pmatrix}
\]
. Using RREF techniques from chapter 7, we get that a basis for the column space is
\[
\begin{align*}
\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \\
\begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \\
\begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}
\end{align*}
\]
which, when we convert these coordinates back into elements of the original vector space \( \mathbb{R}^2 \), gives the answer
\[
\{ (1 \ 1), (1 \ 2), (3 \ 4) \}
\]
2. We could take the transpose of the coordinate column matrices to convert them into row matrices, then find a basis for the row space of
\[
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 2 & 1 & 2 \\
3 & 4 & 5 & 6 \\
7 & 8 & 9 & 10
\end{pmatrix}
\]
Using RREF techniques from chapter 7, we get that a basis for the row space is
\[
\{ (1 \ 0 \ 0 \ -1), (0 \ 1 \ 0 \ 1), (0 \ 0 \ 1 \ 1) \}.
\]
When we take the transpose of these elements (to covert them back into column matrices), and then convert these column matrices back into elements of the original vector space \( \mathbb{R}^2 \), we get the answer
\[
\{ (0 \ 0), (0 \ 1), (0 \ 1) \}
\]
Remember that in general, a vector space will have many different bases. Therefore, it should not bother us that our two answers are not the same – both answers are correct!

Day 2: Linear Transformations

Last class, we saw that having a basis allowed us to represent vectors as column matrices of scalars. There is a special type of function between vector spaces \( T : V \to W \) called a linear transformation, and it turns out that having a basis for \( V \) and \( W \) allows us to represent each linear transformation \( T : V \to W \) by a matrix.

Definition: Let \( V, W \) be vector spaces over the same field \( F \). A linear transformation from \( V \) to \( W \) is a function
\[
T : V \to W
\]
that satisfies the properties
\begin{itemize}
  \item \( T(v_1 + v_2) = T(v_1) + T(v_2) \) for all \( v_1, v_2 \in V \).
  \item \( T(c \cdot v_1) = c \cdot T(v_1) \) for all \( v \in V \) and \( c \in F \).
\end{itemize}
The set of all linear transformations from \( V \) to \( W \) is commonly denoted \( \mathcal{L}(V, W) \)

Examples: 1. Let \( C^\infty \) denote the vector space of all functions from \( \mathbb{R} \) to \( \mathbb{R} \) which can be differentiated repeatedly (any number of times, without ever becoming non-differentiable).
\[
T : C^\infty \to C^\infty \\
f \mapsto f'
\]
is a linear transformation. To verify this, note that
\[ T(f + g) = (f + g)' = f' + g' = T(f) + T(g) \]
\[ T(c \cdot f) = (c \cdot f)' = cf' = c \cdot T(f) \]

2. For any \( n \geq 0 \), the function
\[ T : \mathbb{P}_{n+1} \to \mathbb{P}_n \]
\[ p \mapsto p' \]
is a linear transformation.

3. Let \( A \in \mathbb{M}^{m \times n} \). Then the function
\[ T_A : \mathbb{R}^m \to \mathbb{R}^n \]
\[ x \mapsto Ax \]
is a linear transformation.

4. The function
\[ T : \mathbb{P}_2 \to \mathbb{R}^3 \]
\[ p \mapsto (p(0), p(1), p(2)) \]
is a linear transformation.

Now suppose \( T : V \to W \) is a linear transformation, and you pick bases \( E \) (for \( V \)) and \( F \) (for \( W \)) so that you can represent vectors in \( V \) and \( W \) by column matrices. It turns out that there is a matrix \( A \) that “represents” the transformation \( T \) in the sense that if \( w = T(v) \), then \([w]_F = A[v]_E\). Before we define this matrix and study its properties, let’s work out these details in the context of the second example above.

**Thought experiment** Consider the linear transformation
\[ T : \mathbb{P}_3 \to \mathbb{P}_2 \]
\[ p \mapsto p' \]
The bases \( E = \{x^3, x^2, x, 1\} \) for \( \mathbb{P}_3 \) and \( F = \{x^2, x, 1\} \) for \( \mathbb{P}_2 \) provide a correspondence \( \mathbb{P}_3 \leftrightarrow \mathbb{R}^4 \) and \( \mathbb{P}_2 \leftrightarrow \mathbb{R}^3 \), as described last class. The column matrix \((a, b, c, d)^T \in \mathbb{R}^4\) corresponds to \(ax^3 + bx^2 + cx + d \in \mathbb{P}_2\), which is sent to \(3ax^2 + 2bx + c \in \mathbb{P}_2\) by \( T \). This element of \( \mathbb{P}_2 \) corresponds to \((3a, 2b, c)^T \in \mathbb{R}^3\). In other words, when we represent the vectors in \( \mathbb{P}_3 \) and \( \mathbb{P}_2 \) by their coordinates, the transformation \( T \) becomes the function
\[ \mathbb{R}^4 \to \mathbb{R}^3 \]
\[ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \mapsto \begin{pmatrix} 3a \\ 2b \\ c \end{pmatrix} \]

This function above can be described as left-multiplication by a matrix:
\[ \mathbb{R}^4 \to \mathbb{R}^3 \]
\[ x \mapsto \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} x \]

Using our notation from last class, we can summarize this observation by saying that for all \( v \in \mathbb{P}_3 \),
\[ [T(v)]_F = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} [v]_E \]
In the thought experiment above, the fact that we could represent our linear transformation using a matrix was no coincidence: it turns out that every time you have a linear transformation \( T : V \to W \) between finite-dimensional vector spaces, you can represent it by a matrix (although exactly which matrix it is depends on which bases you pick for each of \( V \) and \( W \)). The following definition and property formalizes this idea.

**Definition** Let \( T : V \to W \) be a linear transformation, let \( E = \{v_1, \ldots, v_n\} \) be a basis for \( V \), and let \( F = \{w_1, \ldots, w_m\} \) be a basis for \( W \). Because \( F \) is a basis, each \( T(v_i) \in W \) can be written uniquely as a linear combination of \( F \). In other words, there are scalars \( a_{ij} \) such that

\[
T(v_1) = a_{11}w_1 + a_{21}w_2 + \cdots + a_{m1}w_m \\
T(v_2) = a_{12}w_1 + a_{22}w_2 + \cdots + a_{m2}w_m \\
\vdots \\
T(v_n) = a_{1n}w_1 + a_{2n}w_2 + \cdots + a_{mn}w_m
\]

The **matrix of \( T \) with respect to \( E \) and \( F \)** is the matrix \([T]_E^F\) obtained by assembling these \( a_{ij} \) into a matrix.

\[
[T]_E^F = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}
\]

**Proposition** Let \( T : V \to W \) be a linear transformation, let \( E = \{v_1, \ldots, v_n\} \) be a basis for \( V \), and let \( F = \{w_1, \ldots, w_m\} \) be a basis for \( W \). Then for every \( v \in V \),

\[
[T(v)]_F = [T]_E^F[v]_E
\]

In fact, \([T]_E^F\) is the unique matrix with this property.\(^2\)

**Proof** Let \( v \in V \). Then \( v = \lambda_1 v_1 + \cdots + \lambda_n v_n \) for some scalars \( \lambda_i \). We’ll expand the left-hand side and right-hand side of the equation separately, to see that they’re equal. The left-hand side gives

\[
T(v) = T(\lambda_1 v_1 + \cdots + \lambda_n v_n) \\
= T(\lambda_1 v_1) + \cdots + T(\lambda_n v_n) \\
= \lambda_1(a_{11}w_1 + a_{21}w_2 + \cdots + a_{m1}w_m) + \cdots + \lambda_n(a_{1n}w_1 + a_{2n}w_2 + \cdots + a_{mn}w_m)
\]

so

\[
[T(v)]_F = \begin{pmatrix} \lambda_1 a_{11} + \lambda_2 a_{12} + \cdots + \lambda_n a_{1n} \\ \lambda_1 a_{21} + \lambda_2 a_{22} + \cdots + \lambda_n a_{2n} \\ \vdots \\ \lambda_1 a_{m1} + \lambda_2 a_{m2} + \cdots + \lambda_n a_{mn} \end{pmatrix}
\]

The right-hand side is

\[
\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} = \begin{pmatrix} \lambda_1 a_{11} + \lambda_2 a_{21} + \cdots + \lambda_n a_{m1} \\ \lambda_1 a_{12} + \lambda_2 a_{22} + \cdots + \lambda_n a_{m2} \\ \vdots \\ \lambda_1 a_{1n} + \lambda_2 a_{2n} + \cdots + \lambda_n a_{mn} \end{pmatrix}
\]

which agrees with the left-hand side. To show that this matrix is the *only* matrix satisfying this condition, let \( A \) be any matrix satisfying the condition. When \( v = v_i \), the condition becomes

\[
[T(v_i)]_F = A \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \text{ith column of } A \end{pmatrix}
\]

\(^2\)This means that the only matrix \( A \) that satisfies the equation \([T(v)]_F = A[v]_E \) for all \( v \in V \) is the matrix \( A = [T]_E^F \).
so we see that for every $i$, the $i^{th}$ column of $A$ must be equal to the coordinates for $T(v_i)$ in the basis $F$. But this is precisely the definition of $[T]_F^E$, so $A = [T]_F^E$.

In the proof of the proposition, we used the fact that $T$ is a linear transformation, not just any function between two vector spaces. For example, you can check that the map $P_3 \to P_6$ that multiplies a polynomial by itself is not a linear transformation. In this case, it is not possible to “represent” it by a matrix.

## Day 3: Changing Bases

Let $V$ be a vector space of dimension $n$ over a field $\mathbb{F}$. So far, we have learned two ways that having a basis for $V$ can help us turn questions about $V$ into questions about matrices of numbers.

1. A basis $B$ provides a correspondence between vectors in $V$ and column matrices in $\mathbb{F}$ by associating to each $v \in V$ its coordinates in $B$.

2. Let $T : V \to W$ be a linear transformation between vector spaces. Picking bases $E$ (for $V$) and $F$ (for $W$) provides a way to convert $T$ into a matrix $[T]^F_E$ that “represents” the transformation $T$ in the sense that

$$ [T(v)]_F = [T]^F_E [v]_E $$

(i.e. $[T]^F_E$ is the matrix with the property that whenever $T$ takes the vector $v$ to the vector $w$, then $[T]^F_E$ will take the coordinates of $v$ to the coordinates of $w$)

Sometimes, we have two different bases $B_1, B_2$ for the same vector space $V$. In this case, there is a matrix called the change of basis matrix that helps you convert from coordinates written using $B_1$ into coordinates written using $B_2$.

**Definition** Let $B_1 = \{v_1, \ldots, v_n\}$ and $B_2 = \{w_1, \ldots, w_n\}$ be bases for the same vector space $V$. Because $B_2$ is a basis, we can write each $v_i$ as a linear combination of $\{w_1, \ldots, w_n\}$. That is, there are scalars $p_{ij}$ such that

$$ v_1 = p_{11}w_1 + p_{21}w_2 + \cdots + p_{n1}w_n $$
$$ v_2 = p_{12}w_1 + p_{22}w_2 + \cdots + p_{n2}w_n $$
$$ \vdots $$
$$ v_n = p_{1n}w_1 + p_{2n}w_2 + \cdots + p_{nn}w_n $$

The **change of basis matrix** from $B_1$ to $B_2$ is the matrix $P_{B_1 \to B_2}$ obtained by assembling these $p_{ij}$ into a matrix. That is,

$$ P_{B_1 \to B_2} = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix} $$

**Proposition** Let $B_1 = \{v_1, \ldots, v_n\}$ and $B_2 = \{w_1, \ldots, w_n\}$ be two bases for $V$. Then for every $v \in V$,

$$ [v]_{B_2} = P_{B_1 \to B_2} [v]_{B_1} $$

In fact, $P_{B_1 \to B_2}$ is the unique matrix with this property.

**Proof** Pick $v \in V$. Because $B_1$ is a basis, there are scalars $c_1, \ldots, c_n$ for which $v = c_1v_1 + \cdots + c_nv_n$. This means that $[v]_{B_1} = (c_1 \ c_2 \ \cdots \ c_n)^T$. If we expand each $v_i$ as a linear combination of the $w_i$’s, then group the terms with the same $w_i$ together, we can find the coordinates for $v$ in the basis $B_2$.

$$ v = c_1(p_{11}w_1 + p_{21}w_2 + \cdots + p_{n1}w_n) + c_2(p_{12}w_1 + \cdots + p_{n2}w_n) + \cdots + c_n(p_{1n}w_1 + \cdots + p_{nn}w_n) $$
$$ = (c_1p_{11} + c_2p_{12} + \cdots + c_np_{1n})w_1 + (c_1p_{21} + \cdots + c_np_{2n})w_2 + \cdots + (c_1p_{nn} + \cdots + c_np_{nn})w_n $$

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3For an example of an application, see assignment 2 question 1
Example

For any Proposition $T$, recall that if we have a linear transformation $L: V \rightarrow W$, and bases $E$ (for $V$) and $F$ (for $W$), we can represent $L$ by a matrix $[L]_E^W$. Let’s study how the matrix $[L]_E^W$ changes when we change the bases $V$ and $W$.

**Proposition** Let $T: V \rightarrow W$ be a linear transformation, let $E_1, E_2$ be two bases for $V$, and let $F_1, F_2$ be two bases for $W$. Then

$$[T]_{E_2}^{F_2} = P_{F_1 \rightarrow F_2} [T]_{E_1}^{F_1} P_{E_2 \rightarrow E_1}$$

**Proof** For any $v \in V$,

$$P_{F_1 \rightarrow F_2} [T]_{E_1}^{F_1} P_{E_2 \rightarrow E_1} [v]_{E_2} = P_{F_1 \rightarrow F_2} [T]_{E_1}^{F_1} [v]_{E_1} = P_{F_1 \rightarrow F_2} [T(v)]_{F_1} = [T(v)]_{F_2}$$

But we know that $[T]_{E_2}^{F_2}$ is the unique matrix with this property, so

$$[T]_{E_2}^{F_2} = P_{F_1 \rightarrow F_2} [T]_{E_1}^{F_1} P_{E_2 \rightarrow E_1}$$

**Example** Consider the linear transformation

$$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$x \mapsto \begin{pmatrix} 5/4 & -3/4 \\ -3/4 & 5/4 \end{pmatrix} x$$
At first glance, it seems strange to apply the tools we’ve discussed this past week to study this transformation. After all, we discussed how picking a basis for a vector space can help us represent our linear transformations with matrices, but in this case, we already have a matrix! In fact, let

\[ E = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\} \]

be the standard basis for \( \mathbb{R}^2 \). If we apply the definitions from this week’s lectures, we find that

\[ [T]_{E}^{E} = \begin{pmatrix} 5/4 & -3/4 \\ -3/4 & 5/4 \end{pmatrix} \]

so by using the standard basis for \( \mathbb{R}^2 \), we find that the “matrix” that represents our transformation is just the same matrix that appears in the definition of the transformation. However, let’s see what happens when we express this transformation in a different basis. Consider the basis

\[ F = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \right\} \]

for \( \mathbb{R}^2 \). Then

\[ P_{F \to E} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad P_{E \to F} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix} \]

so

\[ [T]_{F}^{F} = P_{E \to F} [T]_{E}^{E} P_{F \to E} \]

\[ = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 5/4 & -3/4 \\ -3/4 & 5/4 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \]

\[ = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \]

When we write the transformation as a matrix using the basis \( F \), the matrix is diagonal. Diagonal matrices have certain nice properties; over the second half of the course, we’ll learn some of these properties and we’ll learn why we’d want our linear transformations to be expressed as a diagonal matrix. To give you just a simple example, notice that it’s very easy to multiply diagonal matrices together – you just multiply the corresponding terms on the diagonal (try it if you’re not convinced!). This means that

\[ \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}^{100} = \begin{pmatrix} 2^{100} & 0 \\ 0 & (1/2)^{100} \end{pmatrix} \]

This fact lets us do neat trick like the one below:

**Neat Trick** Let’s calculate

\[ \begin{pmatrix} 5/4 & -3/4 \\ -3/4 & 5/4 \end{pmatrix}^{10000} \]

by hand. Because we know that

\[ \begin{pmatrix} 5/4 & -3/4 \\ -3/4 & 5/4 \end{pmatrix} = P_{F \to E} \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} P_{F \to E}^{-1} \]

it follows that

\[ \begin{pmatrix} 5/4 & -3/4 \\ -3/4 & 5/4 \end{pmatrix}^{10000} = P_{F \to E} \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}^{10000} P_{F \to E}^{-1} P_{F \to E} \]

\[ = P_{F \to E} \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}^{10000} P_{F \to E}^{-1} \]

\[ = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2^{10000} & 0 \\ 0 & (1/2)^{10000} \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix} \]

\[ = \begin{pmatrix} 1/2 (2^{10000} + (1/2)^{10000}) & 1/2 (2^{10000} - (1/2)^{10000}) \\ 1/2 (2^{10000} - (1/2)^{10000}) & 1/2 (2^{10000} + (1/2)^{10000}) \end{pmatrix} \]