It is not knowledge but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment.

KARL FRIEDRICH GAUSS
Letter to Farkas Bolyai
(1808)
Everything You Need To Know
About Linear Equations*

It is one of life’s early and unpleasant lessons that any worthwhile achievement—and most of the unworthy ones—requires a certain measure of effort or agony. This is the reason we endure the often humiliating ritual of dating. Dating, though, hadn’t been an issue for Lorenzo; his marriage to Clarice Orsini of Rome had been arranged. Lorenzo’s mother, who scouted her, described her as having “polite manners, but not as warm as our girls” with a “bust difficult to see, for Romans keep their bust well covered.” Frank talk for a mother, I thought. It was a marriage of convenience but who am I to criticize? Wasn’t my relationship to the same Lorenzo one of convenience as well? I

*But Were Afraid to Learn
didn’t want to be penned up in a prison and he didn’t want to be ignorant of linear algebra. But Lorenzo was realizing that there was an ample effort required and more than a little agony involved in mastering this new-fangled subject.

To his credit, Lorenzo never wavered in his commitment to learning, or at least he never let me see it. I would sometimes catch him in his library working on the exercises I had assigned. He was engrossed or perhaps simply grossed out judging by some of the groans he so audibly released and with each I feared for my position. Lorenzo was the embodiment of the Renaissance man, enlightened, talented and inquisitive, but his heart was devoted to arts and letters not mathematics. What if he lost interest or no longer saw the importance of my lessons? What then for me? Of one thing I was certain: The fact that I had tenure at a university thousands of miles and hundreds of years away wouldn’t help.

Neither would the court mathematician who had taken to hissing at me every time he saw me. The snake. He tried to undermine me at every turn, dismissing my exercises as worthless mental calisthenics. But true to his nature, Lorenzo was open-minded and extended to me the deference of a dedicated and willing student. Nevertheless, I felt the need to make the subject of linear algebra more relevant. Fortunately, we’re now getting to the meat of the matter. All vegetarians may now be excused.

3.1 The Nature of Solutions

We will soon be in a position to explore a surfeit of applications. And it all starts with the solution of

\[ \mathbf{Ax} = \mathbf{b} \]

for \( \mathbf{x} = [x_j] \in \mathbb{R}^m \) where \( \mathbf{A} = [a_{ij}] \in \mathbb{R}^{m \times n} \) and \( \mathbf{b} = [b_i] \in \mathbb{R}^m \). This is the matrix shorthand for a system of linear equations,

\[
\begin{align*}
    a_{11}x_1 + \cdots + a_{1j}x_j + \cdots + a_{1n}x_n &= b_1 \\
    \vdots & \quad \vdots \\
    a_{i1}x_1 + \cdots + a_{ij}x_j + \cdots + a_{in}x_n &= b_i \\
    \vdots & \quad \vdots \\
    a_{m1}x_1 + \cdots + a_{mj}x_j + \cdots + a_{mn}x_n &= b_m
\end{align*}
\]

These are \( m \) equations in \( n \) unknowns.

—Remind me again why I’d want to learn this material, said Lorenzo.
This form of equations can represent a wide variety of systems from the domain of economics to the field of science.

—And shall we undertake to study specific applications?

Oh yes, indeed, I said assuredly.

The simplest example to visualize, however, is the geometric intersection of lines or planes. Now, we’re not talking just about lines in two-dimensional space or planes in three-dimensional space but more generally “hyperplanes” in $n$-dimensional space.

—$n$ dimensions?

Yes.

—I do not see that.

Quite frankly, neither can I. That’s why we use algebraic equations.

### How Many Solutions?

The solution of $Ax = b$, geometrically, amounts to the intersection of lines “hyperplanes” in $\mathbb{R}^n$, in general. However, we know that sometimes there is no solution, as when we have parallel lines, or even an infinity of solutions, as when we consider two planes intersecting in a line in three-dimensional space. We may, in fact, formalize this result.

\[
\begin{bmatrix}
\text{Theorem I.} & \text{The system of linear equations } Ax = b \text{ where } A \in \mathbb{R}^m, \\
& x \in \mathbb{R}^n \text{ and } b \in \mathbb{R}^m \text{ has either (i) no solution or (ii) a unique solution} \\
& \text{or (iii) infinitely many solutions.}
\end{bmatrix}
\]

—This does not seem very enlightening. It essentially says that the system either has a solution or it doesn’t.

Not exactly. If it does have a solution, then according to this theorem, it has either exactly one or infinitely many. Explicitly precluded is the case of multiple but finite solutions.

**Proof.** We need only show that if two solutions, $x_0$ and $x_1$, say, are distinct, then there are infinitely many solutions. To this end, consider $x_\lambda = x_0 + \lambda(x_1 - x_0)$ where $\lambda \in \mathbb{R}$. Note that when $\lambda = 0$, $x_\lambda = x_0$ and when $\lambda = 1$, $x_\lambda = x_1$. But $x_\lambda$ is a solution for any $\lambda$ because

\[
Ax_\lambda = A[x_0 + \lambda(x_1 - x_0)] = Ax_0 + \lambda(Ax_1 - Ax_0) = b + \lambda(b - b) = b
\]
As there are infinite possibilities for \( \lambda \), there must be infinitely many solutions provided we can show that each \( x_\lambda \) is unique. That is, we also need to prove that \( x_\lambda \neq x_\mu \) for \( \lambda \neq \mu \). This we can do by contraposition.

Assume \( x_\lambda = x_\mu \). Then, \( x_0 + \lambda(x_1 - x_0) = x_0 + \mu(x_1 - x_0) \) and hence \( (\lambda - \mu)(x_1 - x_0) = 0 \). But \( x_1 - x_0 \neq 0 \) because \( x_0 \) and \( x_1 \) are distinct. Therefore, \( \lambda - \mu = 0 \), i.e., \( \lambda = \mu \).

So the numbers of significance are 0, 1 and \( \infty \).

Exactly.

What Form of Solution?

There is a particular version of the general system that is of particular interest. It is

\[
Ax = 0
\]

Such a system of linear equations is called a homogeneous system. By contrast, \( Ax = b \), where \( b \neq 0 \), is known as an inhomogeneous system.

A homogeneous system always has a solution because \( x = 0 \) satisfies \( Ax = 0 \). We refer to this as the trivial solution.

Now the solutions to inhomogeneous systems are closely related to the solutions of the corresponding homogeneous systems.

**Theorem II.** Let \( x_0 \) be a particular solution of \( Ax = b \), where \( A \in m \times n \). Then \( x \) is a solution of \( Ax = b \) if and only if \( x = x_0 + \xi \), where \( \xi \) is a solution of \( Ax = 0 \).

**Proof.** \( \Rightarrow \) Suppose \( x \) is a solution of \( Ax = b \) and consider \( \xi = x - x_0 \). Then

\[
A\xi = A(x - x_0) = Ax - Ax_0 = b - b = 0
\]

Thus \( x = x_0 + \xi \), where \( \xi \) is a solution of \( A\xi = 0 \).

\( \Leftarrow \) Suppose \( x = x_0 + \xi \), where \( \xi \) is a solution of \( A\xi = 0 \). Then

\[
Ax = A(x_0 + \xi) = Ax_0 + A\xi = b + 0 = b
\]

Thus \( x \) is a solution of \( Ax = b \).
This, I may add, is precisely the same approach taken in solving linear differential equations. One obtains a particular solution, as in $x$, and the homogeneous solution, represented by $\xi$. The general solution is the superposition of the two.

### 3.2 Solution of Systems of Linear Equations

We're finally get to the business of solving systems of linear algebraic equations.

—*How do we solve a system of linear equations?* said Lorenzo impatiently.

We might begin with the following theorem.

—*More theorems!* he grumbled under his breath but loud enough to make sure I heard.

$$
\begin{align*}
\text{Theorem III.} & \quad \text{Let } A \in \mathbb{R}^{m \times n} \text{ and } x \in \mathbb{R}^n. \text{ Then } x \text{ is a solution to } Ax = 0 \text{ if and only if } x \text{ is a solution to } \tilde{A}x = 0, \text{ where } \tilde{A} \text{ is the row-reduced echelon form of } A. \text{ Furthermore, } x \text{ is a solution to } Ax = b \text{ if and only if } x \text{ is a solution to } \tilde{A}x = b', \text{ where } b' \text{ is obtained from } b \text{ by performing the same row-reduction operations as applied to } A.
\end{align*}
$$

*Proof.* As $\tilde{A} = EA$, where $E = E_sE_{s-1} \cdots E_1$, represents the sequence of elementary row operations and $E$, we have that if $Ax = 0$, then by premultiplying by $E$ yields $\tilde{A}x = 0$. Conversely, if $\tilde{A}x = 0$, then premultiplying by $E^{-1}$, which must exist owing to the invertibility of each elementary matrix, $Ax = 0$.

Similarly, if $Ax = b$, then by premultiplying by $E$, gives $\tilde{A}x = Eb = b'$. And, conversely, if $\tilde{A}x = b'$, then by premultiplying by $E^{-1}$ leads to $Ax = b$.

In other words, $Ax = 0$ and $\tilde{A}x = 0$ are equivalent systems as are $Ax = b$ and $\tilde{A}x = b'$. The key therefore to solving a system of linear equations is, in general, to row-reduce the augmented matrix $[A \mid b]$. Shall we do an example?

—*Yes, yes,* Lorenzo said, somewhat unsure of what preceded.
Exempli gratia I

Let us solve the system,

\[
\begin{align*}
    x_1 + 2x_2 + 3x_3 + x_4 &= 1 \\
    2x_1 + 3x_2 - x_3 - x_4 &= 1 \\
    x_2 + x_3 - 2x_4 &= 0 \\
    3x_1 + 7x_2 + 4x_3 - 4x_4 &= 2
\end{align*}
\]

(1)

—What does this mean?
Mean?
—Yes, what does this system represent? Does it represent something in the real world or is it idle mathematics?

Well, I wouldn’t want to think of it as idle mathematics. But it doesn’t really stand for anything either; it’s just an example to warm us up although we may think of these equations as “hyperplanes” in \( \mathbb{R}^4 \).

—Hmph. He was manifestly uncomfortable beyond the third dimension, which would make my task in the future all the more difficult. If he thought this was abstract, well... In any case, back to the example.

We form the augmented matrix as

\[
[A | b] = \begin{bmatrix}
    1 & 2 & 3 & 1 & 1 \\
    2 & 3 & -1 & -1 & 1 \\
    0 & 1 & 1 & -2 & 0 \\
    3 & 7 & 4 & -4 & 2
\end{bmatrix}
\]

Performing row reduction gives

\[
[A | b] \rightarrow \begin{bmatrix}
    1 & 2 & 3 & 1 & 1 \\
    0 & -1 & -7 & -3 & -1 \\
    0 & 1 & 1 & -2 & 0 \\
    0 & 1 & -5 & -7 & -1
\end{bmatrix} \rightarrow \begin{bmatrix}
    1 & 2 & 3 & 1 & 1 \\
    0 & 1 & 7 & 3 & 1 \\
    0 & 1 & 1 & -2 & 0 \\
    0 & 1 & -5 & -7 & -1
\end{bmatrix} \rightarrow \begin{bmatrix}
    1 & 2 & 3 & 1 & 1 \\
    0 & 1 & 7 & 3 & 1 \\
    0 & 1 & 1 & -2 & 0 \\
    0 & 1 & -5 & -7 & -1
\end{bmatrix}
\]

\( r_2 - 2r_1 \)

\( r_4 - 3r_1 \)

\( (-1)r_2 \)
The equivalent system, i.e., \( \tilde{A} \tilde{x} = \tilde{b}' \), is

\[
\begin{align*}
\tilde{x}_1 & + \frac{25}{6} \tilde{x}_4 = \frac{5}{6} \\
\tilde{x}_2 & - \frac{17}{6} \tilde{x}_4 = -\frac{1}{6} \\
\tilde{x}_3 & + \frac{5}{6} \tilde{x}_4 = \frac{1}{6}
\end{align*}
\]

(2)

—What’s happened? asked a surprised Lorenzo.

We row-reduced the original systems of linear equations.

—No, no. We had four equations at the beginning. Now we have only three. By my rudimentary arithmetic, we lost one equation. What happened to it?

Ah, the original four equations, we have discovered by row reduction, were not all independent. One might say that there was less information in them than met the eye. However, by row-reducing the system, we have teased out the independent components. I am glossing over this point and I apologize but we shall devote considerable time to studying this issue. For now, we shall have to content ourselves in finding the solution.

Having three equations in four unknowns, we do not have a unique solution. We have \( \tilde{x}_4 \) as a free variable. The solution is therefore of the form

\[
x = \begin{bmatrix}
\frac{5}{6} \\
-\frac{1}{6} \\
\frac{1}{6} \\
0
\end{bmatrix}
+ \tilde{x}_4 \begin{bmatrix}
-\frac{25}{6} \\
\frac{17}{6} \\
-\frac{5}{6} \\
1
\end{bmatrix}
\]
3.2 Solution of Systems of Linear Equations

which is a the equation of a line in \( \mathbb{R}^4 \). So, in accordance with Theorem 1, there is an infinity of solutions. Moreover, as established by Theorem 1, we may identify

\[
\begin{bmatrix}
\frac{5}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
0
\end{bmatrix}, \quad \xi = x_4
\begin{bmatrix}
\frac{-25}{6} \\
\frac{17}{6} \\
\frac{-5}{6} \\
1
\end{bmatrix}
\]

as a particular solution to (1) and the homogeneous solution to \( Ax = 0 \), respectively.

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Do we always get one free variable when we have infinitely many solutions?

asked Lorenzo.

Not at all, I quickly answered. We could have 2 or 3 or 4 or more parameters.

—Really, said Lorenzo, looking closely at my example. After a moment, his eyes seemed to flash. I think I see what is happening. We have one free variable because we have one row of zeros!

I sighed, trying to force myself not to count the number of times I read this erroneous conclusion in high-school algebra textbooks. I said to Lorenzo gently. Let’s consider another example before we draw any conclusion.

**Exempli gratia II**

Consider \( A \in \mathbb{R}^{4 \times 5} \) which has as its row-reduced echelon form,

\[
A = \begin{bmatrix}
1 & 2 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 1 & 3 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Now the solution to \( Ax = 0 \) or, equivalently, \( \tilde{A}x = 0 \) is given by solving

\[
\begin{align*}
x_1 + 2x_2 + x_5 &= 0 \\
x_3 + 2x_5 &= 0 \\
x_4 + 3x_5 &= 0
\end{align*}
\]
which yields
\[
\begin{bmatrix}
-2x_2 - x_5 \\
x_2 \\
-2x_5 \\
-3x_5 \\
x_5 \\
\end{bmatrix} = \begin{bmatrix}
-2 \\
1 \\
0 \\
0 \\
0 \\
\end{bmatrix} + x_5 \begin{bmatrix}
-1 \\
0 \\
-2 \\
-3 \\
1 \\
\end{bmatrix}
\]

We have two parameters in this case, \(x_2\) and \(x_5\).

—I see, acknowledged Lorenzo. We have two parameters but only one zero row.

The rule is that the number of free variables is the number of variables minus the number of equations in the row-reduced echelon form.

Keep in mind, I warned Lorenzo, that this is not a proof; it is merely a conjecture at this point. But we shall in due course prove it rigorously. Lorenzo didn’t seem to be listening; he was gazing studiously at my scribblings. Or perhaps he finally just fell into a catatonic state.

\[\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdott
or, in matrix form,

\[ \mathbf{x} = \mathbf{b}' - \tilde{\mathbf{A}}_f \mathbf{x}_f \]

where \( \mathbf{x}_f = [x_k]_{k \in S^c} \in \mathbb{R}^{n-m_r} \) is the column of free variables only and \( \tilde{\mathbf{A}}_f = [\tilde{a}_{ik}]_{(i,k \in S^c)} \in \mathbb{R}^{m \times (n-m_r)} \) is the \( m \times (n-m_r) \) matrix containing only the columns without leading “1”s.

—I suppose we have the three possibilities still to consider. That is, we can have 0, 1 or infinitely many solutions.

Yes, that’s correct.

—Can we then summarize when we have what?

I do not see why not.

—Neither can I, Lorenzo emphasized.

0—No Solution. We do not have a solution if there are fewer nonzero rows in \( \tilde{\mathbf{A}} \) than in \( [\tilde{\mathbf{A}} | \mathbf{b}] \). In the example above, if we had instead obtained in the end

\[ [\tilde{\mathbf{A}} | \mathbf{b}] = \begin{bmatrix}
1 & 0 & 0 & \frac{25}{6} & 0 \\
0 & 1 & 0 & \frac{17}{6} & 0 \\
0 & 0 & 1 & \frac{5}{6} & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix} \]

we would have been stuck without a solution because the last row essentially states that “0 = 1”...

—Which of course is nonsense unless I have not properly grasped the concept of zero.

But indeed you have, Signore. It is nonsense. Hence, there is no solution in this case. To have a solution, \( \tilde{\mathbf{A}} \) and \( [\tilde{\mathbf{A}} | \mathbf{b}] \) must have the same number of nonzero rows.

1—Unique Solution. Clearly, from (4), we shall have a unique solution when there is no free variable. This can only happen when \( m_r = n \), that is, the number of nonzero rows is also equal to the number of variables.

\( \infty \)—Infinitely Many Solutions. And, finally, if we have at least one free variable, we shall have infinitely many solutions. This requires that \( m_r < n \). We may capsulize the three cases in Table 1.

Forgive me once again for attempting to anticipate the reader’s thoughts but I suspect you may be wondering why I have not introduced the rank of a matrix. (The rank of a \( \infty \) Rank...
matrix, it turns out, is the number of nonzero rows in its row-reduced echelon form.) There a very good answer to that I hope will defer the subject while sustaining your curiosity. It amounts to this: There are myriad ways to introduce an idea and I prefer to introduce the idea of rank in a different context although it’s completely compatible with the number of nonzero rows in the row-reduced echelon matrix.

Gaussian Elimination

Jordan modified what is known as the method of Gaussian elimination. As you may appreciate, putting zeros in the columns above the leading “1”s is a bit of overkill. Thus, in the row-reduction algorithm of §2.2, we change only Step 3. In the first pass, there is no difference between the Gaussian and Gauss-Jordan methods; however, in subsequent passes, only the nonzero entries below the leading “1” are eliminated.

We may write the row echelon matrix \( \hat{A} \) (as opposed to the row-reduced echelon matrix) formed by Gaussian elimination as a series of elementary matrices premultiplying \( A \), except that there are fewer elementary matrices involved than in the construction of \( \tilde{A} \). The row echelon form \( \hat{A} \) is upper-triangular but each nonzero row begins with a leading “1” and it has as many nonzero rows as \( \tilde{A} \).

—This must mean that the solution to \( Ax = b \) is the same as the solution to \( \hat{A}x = b' \), said Lorenzo.

Precisely right. In fact, we can offer a counterpart theorem for Gaussian elimination to Theorem 3.2 for Gauss-Jordan elimination.

Theorem IV. Let \( A \in m \mathbb{R}^n \) and \( x \in n \mathbb{R} \). Then \( x \) is a solution to \( Ax = 0 \) if and only if \( x \) is a solution to \( \hat{A}x = 0 \), where \( \hat{A} \) is the row echelon form of \( A \). Furthermore, \( x \) is a solution to \( Ax = b \) if and only if \( x \) is a solution to \( \hat{A}x = b' \), where \( b' \) is obtained from \( b \) by performing the same elementary row operations as applied to \( A \).

The proof follows the same line as that for Theorem 3.2 and I shall omit it.

<table>
<thead>
<tr>
<th>Type of Solution</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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</tr>
<tr>
<td>1</td>
<td>Unique solution</td>
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<td>( \infty )</td>
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Table 1: Solution of \( Ax = b \)
Back-Substitution. One important difference between Gauss-Jordan and Gaussian elimination is that with the former we can more or less read off the solution \( \mathbf{x} \). In the Gaussian method, however, we need to use back-substitution to extract the solution.

The row echelon form of a system of linear equations, \( \hat{\mathbf{A}} \mathbf{x} = \mathbf{b}' \), can be written as

\[
x_{j_i} + \sum_{k=j_i+1}^{n} \hat{a}_{ik}x_k = \hat{b}_i, \quad i = 1 \cdots m_r
\]

—Wait, interrupted Lorenzo. This looks the same as that for the row-reduced echelon form.

It is different, in general. If you look back at (3), you’ll see that the summation is over only the free variables \( k \in S_c \) and, owing to the row-reduced echelon form, only \( k > j_i \) is actually included in the \( i \)th equation which yields \( x_{j_i} \). But here, in (5), the summation in the \( i \)th equation generally includes all the variables \( x_k \) with \( k > j_i \), not just the free variables.

To solve for \( x_{j_i} \) in (5), we must do so in reverse order,

\[
x_{j_i} = \hat{b}_i - \sum_{k=j_i+1}^{n} \hat{a}_{ik}x_k, \quad i = m_r \cdots 1
\]

That is, we deal with \( i = m_r \) first, followed by \( i = m_r - 1 \), which is then followed by \( i = m_r - 2 \) and so on. In this way, we shall have at each step all the \( x_k \) (including the free variables) on the right-hand side to solve for \( x_{j_i} \).

—Perhaps I need to see an example, said Lorenzo.

Exempli gratia III

Let’s take the previous example and solve the system of linear equations (1) using Gaussian
elimination. The row-reduction steps are as follows:

\[
\begin{bmatrix}
1 & 2 & 3 & 1 & 1 \\
0 & -1 & -7 & -3 & -1 \\
0 & 1 & 1 & -2 & 0 \\
0 & 1 & -5 & -7 & -1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 3 & 1 & 1 \\
0 & 1 & 7 & 3 & 1 \\
0 & 1 & 1 & -2 & 0 \\
0 & 1 & -5 & -7 & -1
\end{bmatrix}

\rightarrow
\begin{bmatrix}
1 & 2 & 3 & 1 & 1 \\
0 & 1 & 7 & 3 & 1 \\
0 & 0 & -6 & -5 & -1 \\
0 & 0 & -12 & -10 & -2
\end{bmatrix}

\rightarrow
\begin{bmatrix}
1 & 2 & 3 & 1 & 1 \\
0 & 1 & 7 & 3 & 1 \\
0 & 0 & 1 & \frac{5}{6} & \frac{1}{6} \\
0 & 0 & -12 & -10 & -2
\end{bmatrix}

\rightarrow
\begin{bmatrix}
1 & 2 & 3 & 1 & 1 \\
0 & 1 & 7 & 3 & 1 \\
0 & 0 & 1 & \frac{5}{6} & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}

= [A|b] = [\hat{A}|b'']

The equivalent system, \( \hat{A}x = b'' \), is thus

\[
\begin{align*}
x_1 + 2x_2 + 3x_3 + x_4 &= 1 \\
x_2 + 7x_3 + 3x_4 &= 1 \\
x_3 + \frac{5}{6}x_4 &= \frac{1}{6}
\end{align*}
\]

---And...so...to solve, we must solve for \( x_1, x_2 \) and \( x_3 \) in reverse order...backwards, Lorenzo said pensively.

Yes. And \( x_4 \), as before, is a free variable. Hence

\[
x_3 = \frac{1}{6} - \frac{5}{6}x_4
\]

and

\[
x_2 = 1 - 7x_3 - 3x_4 = -\frac{1}{6} + \frac{17}{6}x_4
\]
upon inserting the previous solution for $x_3$, and finally

$$x_1 = 1 - 2x_2 - 3x_3 - x_4 = -\frac{1}{6} - \frac{25}{6}x_4$$

upon substituting the previous solutions for $x_2$ and $x_3$. This is, of course, the same solution that we obtained before.

---

—Reviewing the operations in the example, Lorenzo said, I don’t understand, though, why we have to perform back-substitution in the Gaussian method but we don’t in the Gauss-Jordan method.

In the Gaussian procedure, the back-substitution step is explicit. In the Gauss-Jordan method, this is implicit; we are essentially doing the back-substitution as we go along.

—So, what’s the difference between the two approaches?

A fair question. The Gauss-Jordan elimination, for a square matrix ($m = n$), requires roughly $\frac{1}{2}n^3$ multiplications and additions—“flops” in modern computer jargon—but the Gaussian procedure (including back-substitution) requires about $\frac{1}{3}n^3$ multiplications and additions.

—Really?

Really. You do the math.

—That’s quite nasty.

Forgive me, Signore, I said, fearing that I had egregiously offended. It was just an expression.

—No, no, he said, I like your use of irony: an “egregious” offence. He chuckled quietly.

What was he talking about? Then it occurred to me. In Lorenzo’s time, “egregious” meant remarkably good, great. It still does in Italian, I’m quite sure, as in “egregio Signore.” But back then it had a very positive connotation even in English. Why, I remember a passage from the philosopher Hobbes in the seventeenth century referring to himself as “not so egregious a mathematician as you are” meaning “not so excellent a mathematician.” How interesting that the meaning of some words, to use a common mathematical metaphor, can eventually turn $180^\circ$. If Hobbes were writing in the twenty-first century would he have considered himself “not so bad a mathematician as you are.”

In any case, Lorenzo had his mind on something else. He said, I was just thinking that merely doubling the number of equations multiplies the work required to solve by 8!
3.3 Nontrivial Pursuit

As we have seen, we can tell a lot about the solution to a system of linear equations, \( Ax = b \), by looking at its corresponding homogeneous system, \( Ax = 0 \). Let’s explore this a little more, beginning with the following theorem.

**Theorem V.** Let \( A \in \mathbb{R}^{m \times n} \) have \( m_s \) nonzero rows with \( m_s < n \). Then \( Ax = 0 \) has a nontrivial solution.

—You mean we have \( m_s \) equations in \( n \) unknowns? said Lorenzo.
That’s correct.
—And there are more unknowns than equations.
That’s correct as well.
—Well, of course, we should have a nontrivial solution, he said dismissingly.

**Proof.** By Theorem 3.2, \( x \) is a solution to \( Ax = 0 \) if and only if it is a solution to \( \tilde{A}x = 0 \). Now the number of nonzero rows in \( \tilde{A} \) is \( m_r \leq m_s < n \). Accordingly, (3) applies, i.e.,

\[
x_{j_i} + \sum_{k \in S^c} \tilde{a}_{ik} x_k = 0, \quad i = 1 \cdots m_r
\]

Consider \( x_p = 1 \) for some \( p \in S^c \) and \( x_k = 0 \) for all other \( k \in S^c \). Then

\[
x_{j_i} = -\tilde{a}_{ip}, \quad i = 1 \cdots m_r
\]

Thus there exists a nontrivial solution.

We follow with the obvious corollary.
—What? said a startled Lorenzo. Why do you want to buy a garland? Is this some algebraic tradition?

What is the wide, wide world of mathematics was he talking about?
—You’re referring to a “corollary,” aren’t you?

Well, not exactly. The term corollary derives from the Latin corolla, a small garland of flowers. It’s the diminutive of corona or garland.
In Roman times, a corolla was a gift usually bestowed by a visitor on his host. Corollarium is the money paid for a garland. Did you not know that?

Yes. So you see a corollary in mathematics is much more like a corolla than a corollarium because it follows from a theorem with little or no effort; it is a “gift proferred by a theorem.”

—Oh, that kind of corollary, Lorenzo said somewhat disingeniously. But do you mean to say that in addition to propositions, theorems and lemmas—let’s not forget the lemmata—I have to contend with corollaries!

Um, well, yes. But, look, corollaries require little or no proof.

—Lorenzo sighed and sank in his chair. Then picking himself up said, I suppose that’s a consolation of sorts.

Anyway, back to the corollary…

**Corollary.** If \( A \in \mathbb{R}^n \) where \( m < n \), then \( Ax = 0 \) has a nontrivial solution.

Happily, this corollary follows immediately from Theorem 3.3 and requires no proof. Lorenzo was momentarily relieved. Unfortunately, I had to assault him with another theorem.

### Fredholm Alternative

The following result is known as the **Fredholm alternative**, which actually has a more general form but we shall state only for matrices.

**Theorem VI.** **Fredholm Alternative:** If \( A \in \mathbb{R}^n \), then exactly one of the following statement holds:

1. \( Ax = b \) has one and only one solution for each \( b \in \mathbb{R}^n \).
2. \( Ax = 0 \) has a nontrivial solution.

These two statements are mutually exclusive. Clearly, both statements cannot hold for, if they both did, putting \( b = 0 \) in Statement 2, we’d deduce that \( Ax = 0 \) has only the trivial solution contradicting Statement 1. Now we must prove the theorem.
—Hm, this is a new logical construction for me. Lorenzo pondered, I’ve read Aristotle’s Logic but I don’t remember anything like this.

I wasn’t familiar with Aristotle’s Logic, so I couldn’t help Lorenzo with that. But I tried, as I have done so often in my university lectures, to lead to the light, even if it may be a pen light, using the Socratic method. (Socrates was really onto something with that approach.)

I asked Lorenzo, What is the possible logical status of each statement?
—Each statement can be either true or false.
But can they both be true at the same time or both false at the same time?
—No. The theorem says that when Statement 1 is true, Statement 2 must be false and vice versa.

So what is the necessary and sufficient condition that Statement 1 is true?
—Hmmm… The necessary and sufficient condition that Statement 1 is true is that Statement 2 is false because if Statement 1 is true then Statement 2 must be false and if Statement 2 is false then Statement 1 is true.

What’s another way of saying that a statement is false?
—I’m not sure what you mean. But if a statement is false then the negative of the statement is true.

That’s precisely right. Well, then, what must we do to prove the theorem?
—Ah, Lorenzo said brightly, seeing a squirt of light at the end of the tunnel. We need to show that Statement 1 implies the negative of Statement 2 and that Statement 2 implies the negative of Statement 1.

Oh, we were so close. I then said to Lorenzo, That’s completely true but not truly complete.
—You sound like the Sphinx when you talk like that.

You see, I tried to explain, leaving Socrates behind, you have in essence said that we must prove “1 implies ¬2” and “2 implies ¬1.” But these are logically equivalent. In fact, one is the contraposition of the other.

Another way of stating the Fredholm alternative is to say “Statement 1 if and only if the negative Statement 2,” that is,

Ax = b has one and only one solution for each b ∈ \( n\mathbb{R} \) if and only if Ax = 0 has only the trivial solution."

—So this is the Fredholm alternative alternative, he said. Lorenzo liked playing with words. He was after all a poet of some repute. And the proof?
3.3 Nontrivial Pursuit

Proof. $1 \Rightarrow \neg 2$ Given Statement 1, let $b = 0$. The solution then to the homogeneous system, $Ax = 0$, is $x = 0$. Furthermore, owing to the uniqueness of the solution, this must be the only solution.

$\neg 2 \Rightarrow 1$ Consider the row-reduced form of the homogeneous system, $\tilde{A}x = 0$. If only the trivial solution exists for this system then, according to Theorem 3.3, $\tilde{A}$ cannot have fewer nonzero rows than number of variables. As $A$ is square, it follows that $\tilde{A} = 1$. By Theorem 2.1, $A$ must be invertible. Thus, the solution to the inhomogeneous system is $x = A^{-1}b$ which must be unique owing to the uniqueness of the matrix inverse.

Now if we look very closely at the proof, we find that we get more than we asked for, a little gift from Fredholm, if you will.

—I imagine we’re about to be graced with another corollary, Lorenzo said unenthusiastically.

**Corollary.** For $A \in \mathbb{R}^n$, the following statements are equivalent:

1. $Ax = b$ has one and only one solution for each $b \in \mathbb{R}^n$
2. $Ax = 0$ has only the trivial solution
3. $A$ is invertible
4. $Ax = b_0$ has one and only one solution for at least one $b_0 \in \mathbb{R}^n$
5. $Ax = b$ has at least one solution for each $b \in \mathbb{R}^n$

—You call this a corollary? It looks more like a theorem. In fact, it looks like several theorems.

Ah, but this really is a gift, I protested. We just have to unwrap the proof and we shall have our corollary except for a wee bit of ratiocination, I equivocated.

—Always dread that which follows “except,” lamented Lorenzo.

Proof. $1 \Rightarrow 2$ This is implied by the Fredholm alternative.

$2 \Rightarrow 3$ This was proven in the main theorem.

$3 \Rightarrow 4$ As $A$ is invertible, $x_0 = A^{-1}b_0$ and is the one and only solution to $Ax = b_0$ owing to the uniqueness of the matrix inverse.
Suppose that for \( b_0 \in \mathbb{R}^n \), \( Ax = b_0 \) has one and only one solution \( x_0 \).
By the Fredholm alternative, Statement 1 holds, in which case Statement 5 follows immediately, or the homogeneous system \( Ax = 0 \) has a nontrivial solution. But the latter is impossible because, according to Theorem 1, the general solution to \( Ax = b_0 \) can be written as \( x = x_0 + \xi \) where \( \xi \) is a solution to the homogeneous system. However, the solution \( x_0 \) is unique and thus \( x_0 = x_0 + \xi \), which implies that \( \xi = 0 \).

If Statement 5 holds, then there exist \( x_1, x_2 \ldots x_n \) which are solutions corresponding to \( b_1, b_2 \ldots b_n \), respectively. In block form,
\[
Ax = [b_1 \, b_2 \, \cdots \, b_n]
\]
or, briefly,
\[
AX = B
\]
As \( X \in \mathbb{R}^n \) must exist for any \( B \in \mathbb{R}^n \), we may consider \( B = 1 \), in which case \( X \) is the inverse of \( A \). Thus, because \( A \) is invertible, the solution to \( Ax = b \) for each \( b \in \mathbb{R}^n \) is given by \( x = A^{-1}b \) and is unique owing to the uniqueness of the matrix inverse.

—That's all very elegantly abstruse, observed Lorenzo, but can it help us with any of our problems?
I had to find something fast to give this result some credibility.

**Exempli gratia IV**

Do you remember we discovered that
\[
A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]
is invertible if \( ad - bc \neq 0 \).

—Yes, said Lorenzo somewhat impatiently.

Well, we can also show the converse, namely, that \( ad - bc \neq 0 \) if \( A \) is invertible.

—And thus we have a necessary and sufficient condition for the invertibility of a 2×2 matrix!

Let us use a proof by contradiction. Assume that \( ad - bc = 0 \). Then
\[
x = \begin{bmatrix} d \\ -c \end{bmatrix}
\]
is a solution to the homogeneous system $Ax = 0$ for

$$Ax = \begin{bmatrix} ad - bc \\ cd - dc \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

But as $A$ is invertible, we may invoke the above Corollary and impose the condition of Statement 3—that the only solution to the homogeneous system in the trivial solution. Hence, $x = 0$ which means that we must have $c = d = 0$.

—But here we have our contradiction, interrupted Lorenzo, because that would mean the bottom row in $A$ is a zero row and we know that such a matrix is not invertible.

Marvelous, your Magnificence, marvelous.

3.4 Electric Circuits

Lorenzo was seeing a little too much theory, as far as he was concerned. Theorem after theorem, corollary after corollary can be a much for any student. Lorenzo was anxious to see a real application of linear algebra.

Well Signore, I began, you know that item you took from my wrist...

—Yes, I regret that, Lorenzo said apologetically. We seem to have lost it.

Great. Even traveling through time is fraught with the aggravation of losing luggage, despite it being strapped to your person. The item in question was, of course, my wristwatch. I was hoping to have the opportunity of retrieving it at some point. It was one of those digital multifunction watches that could tell time and everything else short of NASDAQ quotations and that was probably only because I didn’t have the latest model.

—Actually, we suspect that someone’s run off with it, he confessed contritely.

Well, it had a power source called a battery that produces an electric current.

—Lorenzo looked at me askance. Lectric current?

No, no, ee-lectric current.

—Ee-lectric current, he repeated. Do you mean like a current of water? Like a stream?

Yes, exactly, I said encouragingly. And the battery is like a pump!
Imagine the water flowing through a pipe with the water being under pressure. Now if the pipe has a constriction in it at some point, then the pressure upstream will be greater than the pressure downstream.

Lorenzo nodded.

**Ohm’s Law**

The same thing happens with an electric current. If the current passes through a certain type of device called a resistor, it meets with resistance just as in the constriction in a pipe and there also occurs a drop in “pressure.” But in electricity we call the pressure voltage. So resistance causes a voltage drop.

These voltage drops obey *Ohm’s Law*:

\[
\text{Ohm’s Law: The voltage drop } v \text{ across a resistor of resistance } R \text{ is given by } \\
v = Ri \\
\text{where } i \text{ is the current through the resistor.}
\]

I was hoping that Lorenzo, like most students, would not ask about Ohm.

—*Who is Ohm?*

Damn! Some students can be so unpredictable. I guess he wasn’t bothered by names used as adjectives, such “Gaussian elimination” or even the “Kronecker delta,” but the possessive clearly indicated that it wasn’t just a mathematical descriptor like “invertible” but that there was an actual person behind it.

Oh, uh, I stammered to Lorenzo, Georg Simon Ohm was...er, will be...um, is (I concluded that the present tense was as good as any other) a Prussian teacher of mathematics and physics.

—*Ah, like yourself?*

Well, I couldn’t exactly say that. Ohm is a very accomplished individual and, besides, he actually has a law named after him.

There was a papable pause as though Lorenzo was wondering why I and not Ohm was teaching him mathematics. One reason was I happened to be available; that and the little matter that Ohm would not be born for another three hundred years.

Let’s look at a simple example. Consider the circuit in Figure 1. The strength of the battery is 20 V. (Voltage is measured in *volts* after Italian physicist Count Alessandro...
Volta, 1745–1827. Yes, a Count no less. I didn’t want to have to explain this to Lorenzo. I could just imagine his reaction: “Count Alessandro Volta? I know no such person and I am acquainted with all the noblemen of Italy!” The resistors are $R_1 = 2 \Omega$ and $R_2 = 3 \Omega$. (Resistance is appropriately measured in ohms.)

—And what is the current?
Well, that’s what we must determine.

—Ah, I see. And this we do by Ohm’s Law.
There’s a little more to it than that. We have two resistors and we don’t know a priori what the voltage across each resistor is.

—Do you mean that we cannot solve this problem?

**Kirchhoff’s Laws**

One shouldn’t be so pessimistic. Nature will of course partition the voltage in some prescribed manner and that should give us hope that we may discover the rule by which the voltages are apportioned. It was Kirchhoff who first postulated this rule, which we may refer to as Kirchhoff’s Loop Law.

![Kirchhoff's Loop Law](image)

Kirchhoff’s Loop Law. The voltage drops in any closed loop must sum to zero, that is,

$$\sum v_k = 0$$

where $v_k$ is the voltage drop across the $k$th component in the loop.

—This Kirchhoff... a Russian, no?
A reasonable presumption given the name but I said I believed that Gustav Robert Kirchhoff is also Prussian. (In fact, he was born in Königsberg, Prussia, which later became Kaliningrad, Russia.)

—Hmph, Lorenzo noted. How do we use this law.

Well, electric current is like a current of water, as we said. So, the current is the same at any point in the circuit. Let this current be $i$. And let the voltage drops across $R_1$ and $R_2$ be $v_1$ and $v_2$, respectively. Accordingly, by Ohm’s Law,

$$v_1 = R_1i, \quad v_2 = R_2i$$

(7)
Now, by Kirchhoff’s Loop Law,

\[ v_0 - v_1 - v_2 = 0 \]  

(8)

where \( v_0 = 20 \) V is the voltage supplied by the battery. Observe that the voltages in Kirchhoff’s Loop Law can be either positive or negative. If we consider the circuit in a clockwise direction, then the voltage of the battery is positive and the voltages across the resistors are negative. Substituting (7) into (8) gives

\[ v_0 - (R_1 + R_2)i = 0 \]

Upon inserting the numbers, we have

\[ i = \frac{v_0}{R_1 + R_2} = \frac{20 \text{ V}}{2 \Omega + 3 \Omega} = 4 \text{ A} \]

(Current is measured in amperes after French mathematician and physicist André Marie Ampère, 1775–1836; 1 A = 1 V/Ω.)

—Very well, interjected Lorenzo. But where are the matrices? I don’t see any linear algebra.

Suppose we have a slightly more complex circuit such as the one in Figure 2. In this case, we can identify 3 loops: \( ABEFA \), \( BCDEB \) and \( ABCDEFA \). We’ll refer to these as Loops 1, 2 and 3, respectively.

—Ah, so we apply Kirchhoff’s Loop Law to each loop.
Let’s do that and see what we get. Setting the strength of the battery to be $v_0 = 30$ V and the voltage across the $k$th resistor as $v_k$,

\[
\text{Loop 1:} \quad v_1 + v_2 + v_3 = 0 \\
\text{Loop 2:} \quad -v_0 - v_3 + v_4 = 0 \\
\text{Loop 3:} \quad -v_0 + v_1 + v_2 + v_4 = 0
\]

—It seems that we have a problem, said Lorenzo. *Four unknowns, $v_1, v_2, v_3, v_4$, but only three equations.*

However, we have Ohm’s Law at our disposal.

—Ah, of course, he said, suddenly remembering.

If we let $i_1$ be the current through the segment $EFAB$ (clockwise being positive), we have

\[v_1 = R_1 i_1, \quad v_2 = R_2 i_1\]

Letting $i_2$ be the current through the segment $EF$ (up being positive), we have

\[v_3 = -R_3 i_2\]

And, finally, with $i_3$ as the current through the segment $BCDE$ (clockwise being positive), we obtain

\[v_4 = R_4 i_3\]
Thus, our equations (9a–c) become

\[
\begin{align*}
\text{Loop 1:} & \quad (R_1 + R_2)i_1 - R_3i_2 = 0 \\
\text{Loop 2:} & \quad + R_3i_2 + R_4i_3 = v_0 \\
\text{Loop 3:} & \quad (R_1 + R_2)i_1 + R_4i_3 = v_0
\end{align*}
\]

—Ah, yes, said a much relieved Lorenzo. *Three equations and three unknowns. We can solve that.*

Yes, but we wouldn’t get a unique solution.

—*What do mean?* rifled back Lorenzo, his demeanor turning to concern.

The last of the equations above is dependent on the other two; it is redundant. If we add (10a) to (10b) we get (10c). This is perhaps even more easily seen in the set of equations (9a–c). So, if we were to row-reduce this system . . .

—*We would get a zero row,* interjected Lorenzo.

Yes.

—*Santo cielo,* said Lorenzo, who cursed only sparingly. *That will once again leave us with more unknowns than equations. How can that be?*

If we take Loop 1 and imagine that we add Loop 2, we end up with Loop 3 because the voltage “drop” across $R_3$ in Loop 1 is in effect canceled by the voltage “step” across $R_3$ in Loop 2. (See Figure 3.)

—Lorenzo appeared startled at such simple and physical interpretation of row reduction. *I see,* he said. *But what do we do now?*

Well, we haven’t taken into account enough about nature. After all, we would expect nature to provide only one solution to this circuit, not infinitely many.

—*What have we failed to take into account,* pressed Lorenzo.

If we imagine electric current again as a stream of water, then the rate at which water enters at any node or junction must equal . . .

—*The rate at which water leaves the node!* exclaimed Lorenzo.

Yes! I enthused. This is Kirchhoff’s Node Law:

\[
\sum_m i_m = 0
\]

where $i_m$ is the current into the node from the $m$th branch.
Thus, considering the node at B,

\[ i_1 + i_2 - i_3 = 0 \]  \hfill (11)

where current into the node is positive and from the node negative. This gives us a third independent equation.

—But what about the node at E?

The node equation at E yields \(-i_1 - i_2 + i_3 = 0\) which is the same as at B. In general, if Kirchhoff’s Node Law is applied at each node, then any one of the resulting equations is redundant.

So we have

\begin{align*}
\text{Loop 1:} & \quad (R_1 + R_2)i_1 - R_3i_2 = 0 \\
\text{Loop 2:} & \quad R_3i_2 + R_4i_3 = v_0 \quad \text{(12a–c)} \\
\text{Node B:} & \quad i_1 + i_2 - i_3 = 0
\end{align*}

or, in matrix form,

\[ \mathbf{Ri} = \mathbf{v} \]

where, using the numerical values,

\[ \mathbf{i} = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} 2 & -4 & 0 \\ 0 & 4 & 2 \\ 1 & 1 & -1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 0 \\ 30 \\ 0 \end{bmatrix} \]
Row-reducing leads to

\[
[R | v] \rightarrow \begin{bmatrix}
1 & 0 & 0 & 6 \\
0 & 1 & 0 & 3 \\
0 & 0 & 1 & 9
\end{bmatrix} = [R | v]
\]

The solution is thus \( i_1 = 6 \, A, \, i_2 = 3 \, A \) and \( i_3 = 9 \, A \).

Exercises

1. Consider the following system of equations:

\[
\begin{align*}
w + x - y + 4z &= 0 \\
2w + x + 3y &= 0 \\
x - 5y + 8z &= 0
\end{align*}
\]

(a) By inspection, determine how many solutions exist for the above system. Explain.

(b) Find the set of all solutions

\[
x = \begin{bmatrix}
w \\
x \\
y \\
z
\end{bmatrix}
\]

for the above system.

2. Let \( A \) be an \( m \times n \) matrix and \( B \) an \( n \times m \) matrix so that \( AB \) is invertible.

(a) If \( m = n \), show that \( BA \) is invertible.

(b) If \( m \neq n \), show that \( BA \) is not invertible. (HINT: First show that \( m < n \).)

3. Let \( A \) be an \( m \times n \) matrix where \( m < n \). Show that \( A \) does not have a “left inverse,” that is, show that there does not exist an \( n \times m \) matrix \( L \) such that \( LA = I \).
4. Does the system of linear equations,
\[
\begin{align*}
  x - 3y + 2z &= 4 \\
  2x + y - z &= 1 \\
  3x - 2y + z &= 5
\end{align*}
\]
have infinitely many solutions?

5. If \( A \) is a \( 3 \times 3 \) matrix and
\[
\begin{bmatrix}
  1 \\
  2 \\
  3
\end{bmatrix}
\]
has only one solution, is \( A \) the product of elementary matrices?

6. If \( A \) is a \( 3 \times 3 \) matrix and
\[
\begin{bmatrix}
  1 \\
  2 \\
  3
\end{bmatrix}
\begin{bmatrix}
  0 \\
  1 \\
  0
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
  1 \\
  3 \\
  2
\end{bmatrix}
\begin{bmatrix}
  0 \\
  1 \\
  0
\end{bmatrix}
\]
is \( A \) invertible?

7. Let \( A \) and \( B \) be \( m \times n \) and \( n \times m \) matrices, respectively. Show that \( AB \) is not invertible if \( m > n \).